# Lesson 31 – Reducible and Irreducible Varieties

In this lesson we establish a structure theorem for affine varieties which is an analog of unique factorization for polynomials. Recall that a polynomial $f \in k[x_1, \dots, x_n]$ is *reducible* if we may factor $f$ nontrivially, that is, if $f = gh$ with neither $g$ nor $h$ a constant polynomial. Otherwise $f$ is *irreducible*. Any polynomial $f \in k[x_1, \dots, x_n]$ may be factored

$$f = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_m^{\alpha_m}$$

where the exponents $\alpha_i$ are positive integers, each polynomial $g_i$ is irreducible and nonconstant, and when $i \neq j$ the polynomials $g_i$ and $g_j$ are not proportional. This factorization is essentially unique as any other such factorization is obtained from this one by permuting the factors and possibly multiplying each polynomial $g_i$ by a constant. The polynomials $g_i$ are *irreducible factors* of $f$.

**Exercise 1** Recall that a hypersurface $V \subseteq k^n$ is an affine variety defined by a single polynomial $f \in k[x_1, \dots, x_n]$, i.e., $V = \mathbf{V}(f)$. Assume $f$ has unique factorization $f = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_m^{\alpha_m}$, what does this factorization say about the hypersurface $\mathbf{V}(f)$?

This decomposition property is shared by general affine varieties.

**Definition** An affine variety $V \subseteq k^n$ is **reducible** if it is the union $V = V_1 \cup V_2$ of proper subvarieties $V_1, V_2 \subsetneq V$. Otherwise, $V$ is **irreducible**. That is, an affine variety $V \subseteq k^n$ is **irreducible** if whenever $V$ is written in the form $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are affine varieties, then either $V_1 = V$ or $V_2 = V$.

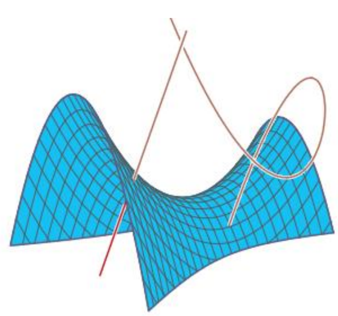

**Figure 1** A reducible variety with 3 components; one is a surface and the other two are curves.

**Exercise 2** Show that the variety $V = \mathbf{V}(xy + z,\ x^2 - x + y^2 + yz)$ is reducible.

The geometric notion of an irreducible variety corresponds to the algebraic notion of a prime ideal. Recall that an ideal $I \subseteq k[x_1, \ldots, x_n]$ is *prime* if whenever $fg \in I$ with $f \notin I$, then we have $g \in I$. Equivalently, if whenever $f, g \notin I$ then $fg \notin I$.

**Proposition 1** An affine variety $V \subseteq k^n$ is irreducible if and only if $\mathbf{I}(V)$ is a prime ideal.

Proof. If $V$ is an affine variety, set $I := \mathbf{I}(V)$.

**Exercise 3** Proving the forward direction, assume $V$ is irreducible and let $f, g \notin I$. Show that $fg \notin I$ and hence $I$ is a prime ideal.

**Exercise 4** Now assume $V$ is reducible. Show that $I$ is not prime.

$$
\begin{array}{ccc}
\{\text{ideals in } k[x_1, x_2, \ldots, x_n]\} & \leftrightarrow & \{\text{subsets in } k^n\} \\
\cup & & \cup \\
\{\text{radical ideals }\} & \leftrightarrow & \{\text{affine varieties}\} \\
\cup & & \cup \\
\{\text{prime ideals}\} & \leftrightarrow & \{\text{irreducible varieties}\}
\end{array}
$$

**Figure 2** A good diagram to keep in mind. The double arrows represent the correspondences **V** and **I**

Before going any further, let's look at some examples.

**Exercise 5** Consider the ideal $I = \langle uw - v^2, u^3 - vw \rangle$. Here's a Groebner basis under the lex order with $w > u > v$.

> $Basis([u \cdot w - v^2, u^3 - v \cdot w], plex(w, u, v));$

$$[-v^3 + u^4, -u^3 + v\,w, u\,w - v^2]$$

a) Show that $u(w^2 - u^2 v) \in I$, but $u \notin I$, $w^2 - u^2 v \notin I$. Hence $I$ is not a prime ideal.

b) Use what you learned in part a) to write $\mathbf{V}(I)$ as a union of two proper subvarieties, $\mathbf{V}(I) = V_1 \cup V_2$.

c) We want to know that the two subvarieties are irreducible. Start with the easy one. Prove $V_1 := V \cap \mathbf{V}(u) = \mathbf{V}(uw - v^2, u^3 - vw, u)$ is irreducible.

d) Finally, prove $V_2 := V \cap \mathbf{V}(w^2 - u^2 v) = \mathbf{V}(uw - v^2, u^3 - vw, w^2 - u^2 v)$ is irreducible, and hence we have decomposed $\mathbf{V}(I)$ into two *irreducible* subvarieties.

The last exercise should have motivated following proposition, which comes in handy when proving varieties are irreducible.

**Proposition 2** If $k$ is an infinite field and $V \subseteq k^n$ is a variety defined parametrically

$$\begin{aligned} x_1 &= f_1(t_1, \dots, t_m), \\ x_2 &= f_2(t_1, \dots, t_m), \\ &\vdots \\ x_n &= f_n(t_1, \dots, t_m), \end{aligned}$$

where $f_1, f_2, \dots, f_n$ are polynomials in $k[t_1, \dots, t_m]$, then $V$ is irreducible.

*Proof.* (See details in textbook; the main idea in the proof was illustrated in the previous example.)