Lesson 32 – Unique Factorization for Varieties

In previous lessons we have seen examples of varieties with one, two, and three irreducible components. Taking products of distinct irreducible polynomials (or dually, unions of distinct hypersurfaces), gives varieties having any *finite* number of irreducible components. As we see next, this is all that can occur...

Theorem 1 (Existence of Decomposition) Any affine variety $V \subseteq k^n$ can be written as a finite union

$$V = V_1 \cup V_2 \cup \cdots \cup V_m,$$

where each V_i is an irreducible subvariety.

So Proposition 1 says any affine variety V has a decomposition

$$V = V_1 \cup V_2 \cup \cdots \cup V_m,$$

where each subvariety V_i is irreducible. Note that we may assume that this decomposition is irredundant in that if $i \neq j$ then V_i is not a subvariety of V_j. If we did have $i \neq j$ with $V_i \subseteq V_j$, then we may remove V_i from the decomposition as a finite union of irreducible subvarieties. Next we tackle uniqueness...

Theorem 2 (Unique Decomposition of Varieties) An affine variety V has a *unique* irredundant decomposition as a finite union of irreducible subvarieties

$$V = V_1 \cup V_2 \cup \cdots \cup V_m.$$

We call the subvarieties V_i the *irreducible components* of V.

Remark. The decomposition must be finite in order to get uniqueness. Indeed any variety is the union of all of its points.

Proof.

Exercise 1 Translate Theorem 1 into the language of ideals using the Algebra-Geometry dictionary.

As the next theorem indicates, ideal quotients can be used to describe the prime ideals appearing in the minimal decomposition of a radical ideal. This is Theorem 6 in section 4.6 of your text.

Theorem 3 If k is algebraically closed and I is a proper radical ideal such that

$$I = P_1 \cap P_2 \cap \cdots \cap P_r$$

is its minimal decomposition as an intersection of prime ideals, then the P_i's are precisely the proper prime ideals that occur in the set $\{I: f \mid f \in k[x_1, \dots, x_n]\}$.

Exercise 2 What is the geometric interpretation of Theorem 2?