Lesson 35 – Algorithmic Computations in $k[x_1, x_2, ..., x_n]/I$

Last lesson we defined the coordinate ring $k[V]$ for an affine variety V:

Definition If $V \subset k^n$ is an affine variety, the **coordinate ring** $k[V]$ is the set of polynomial restrictions $f|_V: V \longrightarrow k$ where $f \in k[x_1, x_2, \dots, x_n]$.

and we concluded that

$$k[V] \cong k[x_1, x_2, \dots, x_n]/\mathbf{I}(V).$$

Hence elements in $k[V]$ are equivalence classes, and two polynomials f and g represent the same element in the coordinate ring iff $f - g \in \mathbf{I}(V)$.

Today we present a way to produce "nice" representatives of the equivalence classes in $k[V]$. The representatives are nice because they will simplify computations within the coordinate ring.

Theorem 1 Fix a monomial order on $k[x_1, x_2, \dots, x_n]$ and let $I \subset k[x_1, x_2, \dots, x_n]$ be an ideal. Then $k[x_1, x_2, \dots, x_n]/I$ is isomorphic as a k-vector space to S = Span($x^\alpha : x^\alpha \notin \langle \mathrm{LT}(I) \rangle$). (If $I = \mathbf{I}(V)$ for an affine variety V, then S is the set of "nice" representatives.)

Exercise 1 You are in the library working on your algebraic geometry homework, and you compute a Groebner basis for $I = \langle xy^3 - x^2, x^3y^2 - y\rangle \subseteq \mathbb{C}[x, y]$ using the grlex order with $x > y$. You get

$$G = \{x^3y^2 - y, x^4 - y^2, xy^3 - x^2, y^4 - xy\}.$$

At this point, a secret admirer approaches you and in an attempt to impress you explains that he or she (depending on your preference) can compute Groebner bases in his/her head. You ask him/her to compute a Groebner basis for $I = \langle xy^3 - x^2, x^3y^2 - y\rangle$ using the lex order with $y > x$. Within seconds, your admirer produces the "basis":

$$Admirer's\ G = \{y - x^7,\ x^{10} - x^2\}.$$

Is this somebody worthy of rearing your children or your dogs (whatever your preference)?

In light of Theorem 1, let's revisit an example we saw last lesson.

Exercise 2 Last lesson we saw that the coordinate ring, $k[\ell]$, of a line $\ell: y = mx + b$ is isomorphic to the infinite dimensional k-vector space $k[x]$. Use Theorem 1 to arrive at the same conclusion.

Exercise 3 What is the coordinate ring for the twisted cubic, $\mathbf{V}(x - z^2, y - z^3) \subset k[x, y, z]$?

Theorem 2 (The Finiteness Theorem) Let $V = \mathbf{V}(I) \subseteq k^n$, where k is algebraically closed and fix a monomial ordering in $k[x_1, x_2, \ldots, x_n]$. Then the following statements are equivalent:

(i) V is a finite set.
(ii) I has a Groebner basis G where for all $1 \leq i \leq n$, G has an element whose leading monomial is a power of x_i. (That is, for each i, $1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} \in \langle \mathrm{LT}(I) \rangle$.)
(iii) Only finitely many monomials are not in $\langle \mathrm{LT}(I) \rangle$.
(iv) $k[x_1, x_2, \ldots, x_n]/I$ is finite dimensional vector space over k.

Exercise 4 - True or False? For each of the statements to follow, assume that k is an algebraically closed field.

True False a) The variety introduced in Exercise 1, $\mathbf{V}(I) = \mathbf{V}(xy^3 - x^2, x^3y^2 - y)$, is a finite affine variety.

True False b) If $V = \mathbf{V}(I) \subseteq k^n$ is a finite affine variety, say $V = \{P_1, P_2, , \dots, P_r\}$, then $r \leq \dim_k k[x_1, x_2, \dots, x_n]/I$.

True False c) If $V = \mathbf{V}(I) \subseteq k^n$ is a finite affine variety, say $V = \{P_1, P_2, , \dots, P_r\}$, then I radical implies $r = \dim_k k[x_1, x_2, \dots, x_n]/I$.