Lesson 36 – Classifying Affine Varieties

Morphisms of Affine Varieties Just as an affine variety is given by polynomials, a morphism of affine varieties is also given by polynomials. The simplest example of a morphism of two affine varieties is a polynomial map $F: k^m \longrightarrow k^n$ defined by

$$F(x_1, x_2, \dots, x_m) = \big(f_1(x_1, x_2, \dots, x_m), f_2(x_1, x_2, \dots, x_m), \dots, f_n(x_1, x_2, \dots, x_m)\big),$$

where $f_i \in k[x_1, x_2, \dots, x_m]$ for all $1 \leq i \leq n$.

In general, a morphism of affine varieties is defined as follows:

Definition Let $V \subseteq k^m$ and $W \subseteq k^n$ be affine varieties. A map $\alpha: V \to W$ is a **morphism** of affine varieties (or a **polynomial mapping**) if it is the restriction of a polynomial map on the affine spaces $k^m \longrightarrow k^n$. A morphism $\alpha: V \to W$ is an **isomorphism** if there exists a morphism $\beta: W \to V$ such that and $\alpha \circ \beta = \mathrm{id}_W$ and $\beta \circ \alpha = \mathrm{id}_V$. If $\alpha: V \to W$ is an isomorphism, then we say that V and W are **isomorphic**.

Easy Example: Show that $\mathbf{V}(y - x^2) \subseteq k^2$ is isomorphic to $\mathbf{V}(x)$.

Common Example: Is $\mathbf{V}(y^2 - x^3) \subseteq k^2$ isomorphic to $\mathbf{V}(x)$?

The Pullback Map Just as each affine variety determines a unique k-algebra (its coordinate ring), every morphism of affine varieties determines a unique k-algebra homomorphism between the corresponding k-algebras.

Indeed, given any morphism $\alpha: V \to W$ of affine varieties, there is a naturally induced map of coordinate rings $\alpha^*: k[W] \to k[V]$ defined by $\alpha^*([f]) = f \circ \alpha$. This map is known as the **pullback map**.

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, dashed, "f \circ \alpha"'] & W \arrow[d, "f"] \\ & k \end{tikzcd}$$

Figure 1: The pullback map

Note that the pullback map is arrow-reversing; the direction of the arrow in $\alpha^*: k[W] \to k[V]$ is in the reverse direction as that in $\alpha: V \to W$.

Exercise 1 Consider the morphism of the varieties $\alpha: k^3 \to k^2$, defined by

$$\alpha(x, y, z) = (x^2 y, x - z).$$

What is the pullback map, α^*?

Proposition 1 Let $V \subseteq k^m$ and $W \subseteq k^n$ be varieties, and let $F: V \to W$ be a morphism. Then the map $F^*: k[W] \to k[V]$, defined by $F^*([g]) = [g \circ F]$ is a k-algebra homomorphism.

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Proposition 1 says that a morphism of affine varieties $V \xrightarrow{F} W$ gives rise to a k-algebra homomorphism $k[W] \xrightarrow{F^*} k[V]$ by the pullback. Conversely, we also have the following theorem, which you will prove in Presentation Assignment #7. (Actually, you will prove something slightly stronger!)

Theorem 2 If $\sigma: k[W] \to k[V]$ is a k-algebra homomorphism, then there is a unique polynomial mapping $F: V \to W$ such that $\sigma = F^*$.

Furthermore, the following is true:

Theorem 3 Two varieties are isomorphic iff their coordinate rings are isomorphic. In particular, $\alpha: V \to W$ is an isomorphism of varieties iff $\alpha^*: k[W] \to k[V]$ is an isomorphism of the corresponding k-algebras.

Theorem 3 is useful in determining whether or not a morphism of varieties is an isomorphism, as the example below will illustrate.

Common Example Revisited Consider, once again, the morphism $\alpha: k \longrightarrow \mathbf{V}(y^2 - x^3) \subseteq k^2$, defined by $\alpha(t) = (t^2, t^3)$.

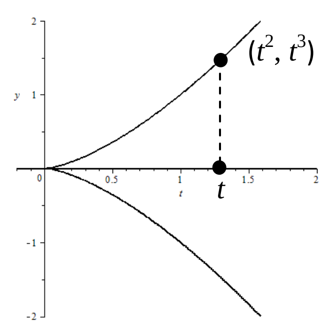

Figure 2: $\mathbf{V}(y^2 - x^3) \subseteq k^2$.

Use the pullback map to prove that α is not an isomorphism of varieties.

Exercise 2 - Isomorphism or not?

a) Consider $\mathcal{C}_1: x^2 + y^2 - 1$ and $\mathcal{C}_2: x^2 + y^2 - 2$ and define $\alpha: \mathcal{C}_1 \to \mathcal{C}_2$ by $\alpha(x, y) = (x + y, x - y)$. Is α an isomorphism of affine varieties or not?

b) Define $\alpha: k^3 \to k^3$ by $\alpha(x, y, z) = (x + 1, 4y + 2z - 2, -x + 2y + z)$. Is α an isomorphism of affine varieties or not?

Exercise 3 - True or False:

a) If $f(x, y) \in k[x, y]$, then $V = \mathbf{V}(z - f(x, y)) \subseteq k^3$ is isomorphic as a variety to k^2.

b) A morphism of affine varieties sends subvarieties to subvarieties.