Lesson 37 - The Variety of a Monomial Ideal

In the final week of the semester we will examine the Hilbert function, a function that is constructed to compute the dimension of a variety. The first step in this process will be to better understand varieties defined monomial ideals. First we’ll need a couple definitions.

**Definition.** A vector space $V \subseteq k^n$ defined by setting some subset of the variables $x_1, x_2, ..., x_n$ equal to zero is called a **coordinate subspace**. A vector space obtained by setting one of the variables equal to zero is called a **coordinate hyperplane**.

**Theorem 1 (Characterization of Varieties of Monomial Ideals)** If $I = \langle x^\alpha : \alpha \in A \rangle \subseteq k[x_1, x_2, ..., x_n]$ is a monomial ideal (so $A \subseteq \mathbb{Z}_{\geq 0}^n$), then the variety $V(I)$ is a finite union of coordinate subspaces of $k^n$.

Proof.

**UPSHOT:** If $I \subseteq k[x_1, x_2, ..., x_n]$ is a monomial ideal, then we have a decomposition:

$$V(I) = V_1 \cup V_2 \cup \cdots \cup V_r,$$

where each $V_i$ has the form $V_j = V(x_i : i \notin \{i_1, i_2, ..., i_j\})$ where $\{i_1, i_2, ..., i_j\}$ is a subset of $\{1, 2, ..., n\}$. (In other words, each $V_j$ is defined by setting a subset of the variables $x_1, x_2, ..., x_n$ equal to zero.) Note that we may assume $V_i \not\subseteq V_j$ for $i \neq j$ to get a minimal decomposition.

**The Dimension of a Variety of a Monomial Ideal**

While our ultimate goal is to compute the dimension of an affine variety in general, it is very easy to compute the dimension of a monomial ideal. So this will be our starting point.

**A Provisional Definition** If a variety $V$ is the union of a finite number of linear subspaces, then the **dimension** of $V$, denoted by $\dim V$, is the largest of the dimensions of the subspaces.

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1 It is not too hard to see that this provisional definition coincides with our earlier definition of dimension (introduced in an earlier homework assignment), when the affine variety is a finite union of linear subspaces.
**Exercise 1** What is the dimension of $V(I) \subseteq \mathbb{C}^4$, where $I = \langle wx^2y, xyz^3, wz^5 \rangle \subseteq \mathbb{C}[x, y, z, w]$?

Could we have somehow computed the dimension above without computing a decomposition? The answer is yes, as the following Proposition indicates.

**Proposition 2** If $I = \langle x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, define

$$M_j = \{x_i: x_i \text{ divides } x^{\alpha_j}\}$$

the set of variables appearing in the $j$th monomial, and consider all subsets $J \subseteq \{x_1, x_2, \ldots, x_n\}$. Then

$$\dim V(I) = n - \min \{|J|: J \cap M_j \neq \emptyset \text{ for all } 1 \leq j \leq r\}$$

(That is, the dimension of $I$ is the smallest number of “zeroed” variables required to guarantee each monomial is zero.) See page 441 of your textbook for the proof.

**Exercise 2** For this exercise, assume that $H_{x_{t_1}x_{t_2} \cdots x_{t_r}}$ denotes the coordinate subspace defined by setting $x_{t_1} = x_{t_2} = \cdots = x_{t_r} = 0$.

a) If $I = \langle x^{\alpha}: \alpha \in A \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, then $V(I)$ contains $H_{x_i}$ as a subspace if ________________.

b) If $I = \langle x^{\alpha}: \alpha \in A \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, then $V(I)$ contains $H_{x_i x_j}$ as a subspace if ________________.

c) If $I = \langle x^{\alpha}: \alpha \in A \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, then $V(I)$ contains $H_{x_{t_1}x_{t_2} \cdots x_{t_r}}$ as a subspace if ________________.
Exercise 3 True or False: If \( I = \langle x^{a_1}, x^{a_2}, \ldots, x^{a_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n] \) is a monomial ideal, it is possible for \( \dim V(I) = 0 \).

Finally, for a good review for Exam III, let’s examine the radical of a monomial ideal. We’ll use some notation that appeared early in the semester.

**Notation:** If \( x^\alpha = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} \in k[x_1, x_2, \ldots, x_n] \), then
\[
\text{rad}(x^\alpha) = x_1^{\delta(a_1)}x_2^{\delta(a_2)}\ldots x_n^{\delta(a_n)},
\]
where \( \delta: \mathbb{Z}_{\geq 0} \to \{0,1\} \) is defined by
\[
\delta(m) = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m > 0 \end{cases}
\]
In other words, \( \text{rad}(x^\alpha) \) is simply the product of the variables dividing \( x^\alpha \), so we are simply creating “squarefree” monomials...

**Definition** A monomial \( x^\alpha = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} \) is said to be **squarefree** if \( a_i \leq 1 \) for \( i = 1, \ldots, n \). A monomial ideal \( I = \langle x^{a_1}, x^{a_2}, \ldots, x^{a_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n] \) is a **squarefree monomial** if all monomials \( x^{a_i}, 1 \leq i \leq r \), are squarefree.

**Theorem 3** If \( I = \langle x^{a_1}, x^{a_2}, \ldots, x^{a_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n] \) is a monomial ideal, then \( \sqrt{I} = \langle \text{rad}(x^{a_1}), \text{rad}(x^{a_2}), \ldots, \text{rad}(x^{a_r}) \rangle \).

**Proof.** The proof will follow quickly from the following lemma.

**Lemma** Any squarefree monomial ideal is a finite intersection of monomial prime ideals.

**Exercise 4** Assuming the lemma, prove Theorem 3.
**Exercise 5** Now let’s prove the lemma...

a) Show that any squarefree principal monomial ideal in $k[x_1, x_2, \ldots, x_n]$ is a finite intersection of monomial prime ideals.

b) Now show that any squarefree monomial ideal in $k[x_1, x_2, \ldots, x_n]$ is a finite intersection of monomial prime ideals.