Lesson 37 - The Variety of a Monomial Ideal

In the final week of the semester we will examine the Hilbert function, a function that is constructed to compute the dimension of a variety. The first step in this process will be to better understand varieties defined monomial ideals. First we'll need a couple definitions.

Definition. A vector space $V \subseteq k^n$defined by setting some subset of the variables $x_1, x_2, \ldots, x_n$ equal to zero is called a **coordinate subspace**. A vector space obtained by setting one of the variables equal to zero is called a **coordinate hyperplane**.

Theorem 1 (Characterization of Varieties of Monomial Ideals) If $I = \langle x^\alpha : \alpha \in A \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal (so $A \subseteq \mathbb{Z}^n_{\geq 0}$), then the variety $\mathbf{V}(I)$ is a finite union of coordinate subspaces of k^n.

Proof.

UPSHOT: If $I \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, then we have a decomposition:

$$\mathbf{V}(I) = V_1 \cup V_2 \cup \cdots \cup V_r,$$

where each V_i has the form $V_j = \mathbf{V}\big(x_i : i \notin \{i_1, i_2, \ldots, i_j\}\big)$ where $\{i_1, i_2, \ldots, i_j\}$ is a subset of $\{1, 2, \ldots, n\}$. (In other words, each V_j is defined by setting a subset of the variables $x_1, x_2, \ldots, x_n$ equal to zero.) Note that we may assume $V_i \not\subseteq V_j$ for $i \neq j$ to get a minimal decomposition.

The Dimension of a Variety of a Monomial Ideal

While our ultimate goal is to compute the dimension of an affine variety in general, it is very easy to compute the dimension of a monomial ideal. So this will be our starting point.

A Provisional Definition[1] If a variety V is the union of a finite number of linear subspaces, then the **dimension** of V, denoted by dim V, is the largest of the dimensions of the subspaces.

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[1] It is not too hard to see that this provisional definition coincides with our earlier definition of dimension (introduced in an earlier homework assignment), when the affine variety is a finite union of linear subspaces.

Exercise 1 What is the dimension of $\mathbf{V}(I) \subseteq \mathbb{C}^4$, where $I = \langle wx^2y, xyz^3, wz^5 \rangle \subseteq \mathbb{C}[x, y, z, w]$?

Could we have somehow computed the dimension above without computing a decomposition? The answer is yes, as the following Proposition indicates.

Proposition 2 If $I = \langle x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_r} \rangle \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal, define

$$M_j = \{x_i\colon\ x_i \text{ divides } x^{\alpha_j}\} = \text{the set of variables appearing in the } j\text{th monomial},$$

and consider all subsets $J \subseteq \{x_1, x_2, \dots, x_n\}$. Then

$$\dim \mathbf{V}(I) = n - \min\{|J|\colon\ J \cap M_j \neq \emptyset \text{ for all } 1 \leq j \leq r\}$$

(That is, the dimension of is the smallest number of "zeroed" variables required to guarantee each monomial is zero.) See page 441 of your textbook for the proof.

Exercise 2 For this exercise, assume that $H_{x_{i_1}x_{i_2}\cdots x_{i_r}}$ denotes the coordinate subspace defined by setting $x_{i_1} = x_{i_2} = \cdots = x_{i_r} = 0$.

a) If $I = \langle x^\alpha\colon \alpha \in A \rangle \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal, then $\mathbf{V}(I)$ contains H_{x_i} as a subspace if ______________________________________.
b) If $I = \langle x^\alpha\colon \alpha \in A \rangle \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal, then $\mathbf{V}(I)$ contains $H_{x_ix_j}$ as a subspace if ______________________________________.
c) If $I = \langle x^\alpha\colon \alpha \in A \rangle \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal, then $\mathbf{V}(I)$ contains $H_{x_{i_1}x_{i_2}\cdots x_{i_r}}$ as a subspace if ______________________________________.

Exercise 3 True or False: If $I = \langle x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, it is possible for $\dim \mathbf{V}(I) = 0$.

Finally, for a **good review for Exam III**, let's examine the radical of a monomial ideal. We'll use some notation that appeared early in the semester.

Notation: If $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in k[x_1, x_2, \ldots, x_n]$, then

$$\mathrm{rad}(x^\alpha) = x_1^{\delta(a_1)} x_2^{\delta(a_2)} \cdots x_n^{\delta(a_n)},$$

where $\delta: \mathbb{Z}_{\geq 0} \to \{0,1\}$ is defined by $\delta(m) = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m > 0 \end{cases}$.

In other words, $\mathrm{rad}(x^\alpha)$ is simply the product of the variables dividing x^α, so we are simply creating "squarefree" monomials...

Definition A monomial $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is said to be **squarefree** if $a_i \leq 1$ for $i = 1, \ldots, n$. A monomial ideal $I = \langle x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a **squarefree monomial** if all monomials x^{α_i}, $1 \leq i \leq r$, are squarefree.

Theorem 3 If $I = \langle x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_r} \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal, then $\sqrt{I} = \langle \mathrm{rad}(x^{\alpha_1}), \mathrm{rad}(x^{\alpha_2}), \ldots, \mathrm{rad}(x^{\alpha_r}) \rangle$.

Proof. The proof will follow quickly from the following lemma.

Lemma Any squarefree monomial ideal is a finite intersection of monomial prime ideals.

Exercise 4 Assuming the lemma, prove Theorem 3.

Exercise 5 Now let's prove the lemma...

a) Show that any squarefree principal monomial ideal in $k[x_1, x_2, \ldots, x_n]$ is a finite intersection of monomial prime ideals.

b) Now show that any squarefree monomial ideal in $k[x_1, x_2, \ldots, x_n]$ is a finite intersection of monomial prime ideals.