Lesson 40 – Introducing the Hilbert Function

Today we introduce the Hilbert Function of an ideal I, a function designed to compute the dimension of a variety by counting the monomials in the complement of the ideal. We start with a couple of definitions.

Definition If I is an ideal in $k[x_1, x_2, , \dots, x_n]$, then we define $\boldsymbol{k[x_1, x_2, \dots, x_n]_{\leq s}}$ to be the set of polynomials in $k[x_1, x_2, , \dots, x_n]$ of total degree $\leq s$ and $\boldsymbol{I_{\leq s}}$ is the set of polynomials in I of total degree $\leq s$. That is,

$$k[x_1, x_2, , \dots, x_n]_{\leq s} = \{f \in k[x_1, x_2, , \dots, x_n] \colon \text{totaldeg}\,(f) \leq s\}$$
$$I_{\leq s} = I \cap k[x_1, x_2, , \dots, x_n]_{\leq s} = \{\, f \in I \colon \text{totaldeg}\,(f) \leq s\}$$

Both $k[x_1, x_2, , \dots, x_n]_{\leq s}$ and $I_{\leq s}$ are vector spaces over k and, clearly, $I_{\leq s}$ is a vector subspace of $k[x_1, x_2, , \dots, x_n]_{\leq s}$. We are ready to define the Hilbert function.

Definition. Let I be an ideal in $k[x_1, x_2, , \dots, x_n]$, and let $I_{\leq s}$ be the space of elements of I of degree at most s. The (affine) **Hilbert function** $HF_I(s)$ of I is defined to be the dimension of $k[x_1, x_2, , \dots, x_n]_{\leq s}/I_{\leq s}$ as a vector space over k. That is,

$$HF_I(s) \stackrel{\text{def}}{=} \dim\,(k[x_1, x_2, \dots, x_n]_{\leq s}/I_{\leq s}).$$

Exercise 1 In fact, $HF_I(s) = \dim\,(k[x_1, x_2, \dots, x_n]_{\leq s}) - \dim\,(I_{\leq s})$. Why?

Exercise 2 Compute the Hilbert function of the zero ideal in $k[x_1, x_2, , \dots, x_n]$.

What is the general form of the Hilbert function? Before tackling this question, we'll focus on monomial ideals first (a common theme in this course!). Monomial ideals are easier to understand.

I. The Hilbert Function of a Monomial Ideal

Exercise 3 Compute the Hilbert function of the ideal $I = \langle x^3y, x^2y^4\rangle$ in $k[x,y]$.

Exercise 4 Compute the Hilbert function of the ideal $I = \langle x^3, x^2y^2, y^4\rangle$ in $k[x,y]$.

The above example illustrated how to compute the Hilbert function for a *monomial ideal*. We will examine such computations more systematically next lesson. Such efforts will pay off because of the next theorem, which is due to MacCaulay (the thesis advisor of J. E. Littlewood).

Theorem (MaCaulay) Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and let > be a graded order[1] on $k[x_1, \dots, x_n]$. Then the monomial ideal $\langle \mathrm{LT}(I)\rangle$ has the same affine Hilbert function as I.

The proof of this theorem will follow quickly from a lemma. If $I \subseteq k[x_1, \dots, x_n]$ is an ideal, then we define the set of leading monomials of total degree $\leq s$ by

$$\mathrm{LM}(I_{\leq s}) = \{\mathrm{LM}(f) : f \in I_{\leq s}\} = \{\mathrm{LM}(f_1), \mathrm{LM}(f_2), \dots, \mathrm{LM}(f_m)\}.^{2}$$

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[1] A graded order on $k[x_1, \dots, x_n]$ is a monomial order satisfying $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$.
[2] The set is clearly finite, because there are only finitely many monomials in $k[x_1, \dots, x_n]$ of total degree $\leq s$.

II. The Proof of MaCaulay's Theorem

Lemma If $I \subseteq k[x_1, \dots, x_n]$ is an ideal and < is a graded order on $k[x_1, \dots, x_n]$, then $I_{\leq s}$ has the same dimension as $\langle \mathrm{LT}(I) \rangle_{\leq s}$.

The next two exercises will prove this lemma.

Exercise 5 Assume $I \subseteq k[x_1, \dots, x_n]$ is an ideal and assume

$$\mathrm{LM}(I_{\leq s}) = \{\mathrm{LM}(f_1), \mathrm{LM}(f_2), \dots, \mathrm{LM}(f_m)\},$$

where $\mathrm{LM}(f_1) > \mathrm{LM}(f_2) > \cdots > \mathrm{LM}(f_m)$. Show that $\mathfrak{B} = \{f_1, f_2, \dots, f_m\}$ is a basis for $I_{\leq s}$ as a vector space over k.

Exercise 6 Assume $I \subseteq k[x_1, \dots, x_n]$ is an ideal and $\mathrm{LM}(I_{\leq s}) = \{\mathrm{LM}(f_1), \mathrm{LM}(f_2), \dots, \mathrm{LM}(f_m)\}$, where $\mathrm{LM}(f_1) > \mathrm{LM}(f_2) > \cdots > \mathrm{LM}(f_m)$. Show that $\mathfrak{B}' = \{\mathrm{LM}(f_1), \mathrm{LM}(f_2), \dots, \mathrm{LM}(f_m)\}$ is a basis for $\langle \mathrm{LT}(I) \rangle_{\leq s}$.

Proof.

We are now ready to prove MaCaulay's theorem:

Theorem Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and let > be a graded order on $k[x_1, \dots, x_n]$. Then the monomial ideal $\langle \mathrm{LT}(I) \rangle$ has the same affine Hilbert function as I.

Proof.

Next lesson we will see that for every monomial ideal J, there exists a non-negative integer t and a univariate polynomial $HP_J \in \mathbb{Q}[s]$ such that

$$HF_J(s) = HP_J(s) \text{ for every } s \geq t \text{ and } \dim(\mathbf{V}(J)) = \deg(HP_J).$$

The polynomial HP_J is known as the "Hilbert Polynomial".

Furthermore, since $HF_J(s) = HF_{\langle \mathrm{LT}(J) \rangle}(s)$, we will be able to compute the dimension, $\dim(\mathbf{V}(J))$, for an arbitrary ideal J (i.e., not necessarily monomial) by counting the monomials in the complement of $\langle \mathrm{LT}(J) \rangle$. We'll end with two examples to illustrate.

Exercise 7

a) Compute the dimension of the affine variety defined by the ideal $I = \langle x^3y^2 + 3x^2y^2 + y^3 + 1 \rangle \subseteq k[x, y]$.

b) Compute the dimension of the affine variety defined by the ideal $I = \langle xz, xy - 1 \rangle \subseteq \mathbb{C}[x, y, z]$.