Lesson 41 – The Complement of a Monomial Ideal

Last lesson we defined the Hilbert Function $HF_I(s)$ of an affine variety I:

$$\begin{aligned} HF_I(s) &\stackrel{\text{def}}{=} \dim\left(k[x_1, x_2, \dots, x_n]_{\le s}/I_{\le s}\right) \\ &= \dim(k[x_1, x_2, \dots, x_n]_{\le s}) - \dim(I_{\le s}) \end{aligned}$$

and showed that $HF_I(s) = HF_{\langle \mathrm{LT}(I)\rangle}(s)$. Today we will see that for large enough s the Hilbert function is given by a polynomial – the "Hilbert polynomial". We start with a quick example...

Exercise 1 Find the Hilbert function of $I = \langle xy^3, x^2y^2\rangle$. Then write $V = \mathbf{V}(xy^3, x^2y^2)$ as a decomposition of coordinate subspaces and use this decomposition to compute the dimension of V.

Definition. Given a monomial ideal $I \subseteq k[x_1, x_2, \dots, x_n]$, we define the "complement exponent set" $C(I)$ by

$$C(I) = \{\alpha \in \mathbb{Z}^n_{\ge 0} : x^\alpha \notin I\}$$

Hence $\alpha = (a_1, a_2, \dots, a_n) \in C(I)$ iff $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \notin I$ (iff $(a_1, a_2, \dots, a_n)$ lives "under the staircase").

Proposition 1 If $I \subseteq k[x_1, x_2, \ldots, x_n]$ is a proper monomial ideal, then the set $C(I) \subseteq \mathbb{Z}_{\geq 0}^n$ can be written as a finite (but not necessarily disjoint) union of translates of coordinate subspaces of $\mathbb{Z}_{\geq 0}^n$.

The next proposition presents a correspondence between coordinate subspaces of $\mathbf{V}(I)$ with coordinate subspaces of $\mathbb{Z}_{\geq 0}^n$ contained in $C(I)$. This result is important because computing dimensions of subspaces of $C(I)$ is easier than computing dimensions of subspaces of $\mathbf{V}(I)$.

Proposition 2 Let $I \subseteq k[x_1, x_2, \ldots, x_n]$ be a proper monomial ideal. Then coordinate subspaces of $\mathbf{V}(I)$ correspond with coordinate subspaces of $\mathbb{Z}_{\geq 0}^n$ contained in $C(I)$. In particular, if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$,

$$\mathbf{V}(x_i : i \notin \{i_1, i_2, \ldots, i_r\}) \subseteq \mathbf{V}(I) \text{ iff } [e_{i_1}, e_{i_2}, \ldots, e_{i_r}] \subseteq C(I)$$

Upshot: If I is a monomial ideal, then the dimension of $\mathbf{V}(I)$ is the dimension of the largest coordinate subspace of $C(I)$. Let's revisit our previous example...

Exercise 2 Given $I = \langle xy^3, x^2y^2 \rangle$, use the decomposition

$$C(I) = [e_1] \cup (e_2 + [e_1]) \cup [e_2] \cup \{(1,2)\}$$

to find the dimension of $\mathbf{V}(I)$.

We are now ready to prove today's main theorem:

Theorem 3 If $I \subseteq k[x_1, x_2, \ldots, x_n]$ is a monomial ideal with $\dim(\mathbf{V}(I)) = d$, then for all s sufficiently large, $HF_I(s)$ is a polynomial of degree d in s.

Sketch of Proof. To determine the number of points in $C(I)^{\leq s}$, we first write $C(I)$ as a finite union of translates (and assume $T_i \neq T_j$ when $i \neq j$):

$$C(I) = T_1 \cup T_2 \cup \cdots \cup T_r$$

where $\dim T_i$ is the same as the dimension of a corresponding coordinate subspace of $\mathbf{V}(I)$. WLOG, we may assume $\dim T_1 \geq \dim T_2 \geq \cdots \geq \dim T_r$, and hence $\dim T_1 = d$, while $\dim T_i \leq d$ for all $2 \leq i \leq r$.

Then $C(I)^{\leq s} = T_1{}^{\leq s} \cup T_2{}^{\leq s} \cup \cdots \cup T_r{}^{\leq s}$, and applying the Inclusion-Exclusion Principle,

$$HF_I(s) = |C(I)^{\leq s}| = \sum_{i=1}^{r} \left|T_i{}^{\leq s}\right| - \sum_{i<j} \left|T_i{}^{\leq s} \cap T_j{}^{\leq s}\right| + \sum_{i<j<k} \left|T_i{}^{\leq s} \cap T_j{}^{\leq s} \cap T_k{}^{\leq s}\right| - \cdots$$

Since T_1 is a translate of a coordinate subspace of $\mathbb{Z}_{\geq 0}^n$ of dimension d, $T_1 = \alpha + [e_{i_1}, e_{i_2}, \ldots, e_{i_d}]$ where $\alpha = \sum_{i \notin \{i_1, i_2, \ldots, i_d\}} a_i\, e_i$. Define $|\alpha| = \sum_{i \notin \{i_1, i_2, \ldots, i_d\}} a_i$.

Exercise 3 Show that for $s > |\alpha|$, $\left|T_1{}^{\leq s}\right| = \binom{d + s - |\alpha|}{s - |\alpha|}$, and deduce that $\left|T_1{}^{\leq s}\right|$ is a polynomial function of s of degree d.

Exercise 4 Similar to the above computation, $\left|T_i^{\leq s}\right| = \binom{m+s-|\alpha|}{s-|\alpha|}$ when $s > |\alpha|$ for translates T_i of dimension m. Use this and the inclusion-exclusion principle (restated below) to deduce that $HF_I(s)$ is a polynomial of degree d in s.

$$HF_I(s) = |C(I)^{\leq s}| = \sum_{i=1}^{r} \left|T_i^{\leq s}\right| - \sum_{i<j} \left|T_i^{\leq s} \cap T_j^{\leq s}\right| + \sum_{i<j<k} \left|T_i^{\leq s} \cap T_j^{\leq s} \cap T_k^{\leq s}\right| - \cdots$$

Definition The **dimension** of an affine variety $V \subseteq k^n$, denoted $\dim V$, is defined to be the degree of the Hilbert polynomial corresponding to the ideal $I = \mathbf{I}(V)$. That is,

$$\dim V \overset{\text{def}}{=} \deg HP_{\mathbf{I}(V)}$$