Lesson 42 – The FINAL Lesson!

Recall the main theorem from last class...

Theorem If $I \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal with $\dim(\mathbf{V}(I)) = d$, then for all s sufficiently large, $HF_I(s)$ is a polynomial of degree d in s.

Furthermore, we showed that the Hilbert function of any ol' ideal I satisfies $HF_I(s) = HF_{\langle \mathrm{LT}(I) \rangle}(s)$, and therefore, $HP_I(s) = HP_{\langle \mathrm{LT}(I) \rangle}(s)$. This allows us to make the following definition:

Definition The **dimension** of an affine variety $V \subseteq k^n$, denoted $\dim V$, is defined to be the degree of the Hilbert polynomial corresponding to the ideal $I = \mathbf{I}(V)$. That is,

$$\dim V \stackrel{\text{def}}{=} \deg HP_{\mathbf{I}(V)}$$

Exercise 1 True or False: If $I \subseteq k[x_1, \dots, x_n]$ is an ideal, then $\dim \mathbf{V}(I) = \deg HP_I$. (That is, can we compute the dimension of a variety V using any ideal defining V?)

Exercise 2 True or False: If $I \subseteq k[x_1, \dots, x_n]$ is an ideal and k is algebraically closed, then $\dim \mathbf{V}(I) = \deg HP_I$.

Proposition 1 If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then the affine Hilbert polynomials of I and $\sqrt{I}$ have the same degree. That is, $\deg HP_I = \deg HP_{\sqrt{I}}$.

Exercise 3 Prove the Proposition in the case where I is a monomial ideal.

Exercise 4 Use the following fact (which you proved in your homework!) to show $\deg HP_I = \deg HP_{\sqrt{I}}$ for *any* ideal I.

Fact from HW: If I_1 and I_2 are two ideals and $I_1 \subseteq I_2$, then $\deg HP_{I_2} \leq \deg HP_{I_1}$

UPSHOT: If k is algebraically closed, then we can compute the dimension of an affine variety using *any* defining ideal by finding the dimension of the largest coordinate subspace of $\mathbf{V}(\langle \mathrm{LT}(I) \rangle)$ (and this can be accomplished by computing the dimension of the largest coordinate subspace of $C(\langle \mathrm{LT}(I) \rangle)$.)

Exercise 5 Compute the dimension of the affine variety defined by the ideal $I = \langle xz, xy - 1 \rangle \subseteq \mathbb{C}[x, y, z]$ using the approach described above.