Group Objects

PCMI Undergraduate Summer School Étude of Questions

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Goal

The aim of this series of questions is to introduce an extrinsic, category-theoretic definition of a group. We will give some examples and explore the idea of group objects in various categories.

Some category theory

This section is hopefully at least a little bit familiar, but essential to everything that follows. The two things that we need to know about are terminal objects and products.

Definition 1. Given a category $\mathbf{C}$, a **terminal object** is an object $\mathbf{1}$ of $\mathbf{C}$ such that, for any other object A of $\mathbf{C}$, there is a unique arrow $A \to \mathbf{1}$.

Question 1:

(a) *Given a set X and any one-element set $\{*\}$, how many functions are there from X to $\{*\}$?*

(b) *What is a terminal object in the category of sets and functions?*

(c) *What is a terminal object in the category of topological spaces and continuous maps?*

(d) *What is a terminal object in the category of groups and group homomorphisms?*

(e) *What is a terminal object in the category of rings and ring homomorphisms? (Rings have identity and ring homomorphisms preserve identity).*

(f) *Let X be a topological space with topology τ. Take the open sets of τ as the objects of a category, with an arrow $U \to V$ if and only if $U \subseteq V$. What is the terminal object in this category?*

In category theory, the basic objects are the objects and arrows, while in set theory, the basic objects are the sets and their elements. Terminal objects give us a bridge between these notions. In **Sets**, an arrow $\mathbf{1} \to X$ picks out an element of the set X. So in a general category $\mathbf{C}$, the generalization of "elements" of an object A are arrows $\mathbf{1} \to A$. Granted, this doesn't always make sense (see part (f) above), but it's often useful nonetheless.

Question 2:*Show that any two terminal objects are isomorphic. Specifically, if* $\mathbf{1}$ *and* $\mathbf{1}'$ *are both terminal objects, show that there are arrows* $f\colon \mathbf{1} \to \mathbf{1}'$ *and* $g\colon \mathbf{1}' \to \mathbf{1}$ *such that* $g \circ f = \mathrm{id}_{\mathbf{1}}$ *and* $f \circ g = \mathrm{id}_{\mathbf{1}'}$. *This is what it means for two objects to be isomorphic in a category.*

Now that we have terminal objects under our belts, let's move on to product objects. This definition simultaneously generalizes the idea of the Cartesian product of sets, the product of groups, and the intersection of open sets in a topological space.

Definition 2. Given a category $\mathbf{C}$, and two objects A and B, the **product** of A and B is an object P, together with maps $\pi_A\colon P \to A$ and $\pi_B\colon P \to B$,

$$A \xleftarrow{\pi_A} P \xrightarrow{\pi_B} B \ ,$$

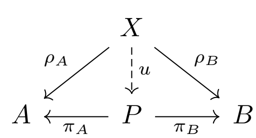
such that for any other object X with maps $\rho_A\colon X \to A$ and $\rho_B\colon X \to B$,

$$A \xleftarrow{\rho_A} X \xrightarrow{\rho_B} B$$

we get the following two things:

(a) **existence:** there is an arrow $u\colon X \to P$ such that $\rho_A = \pi_A \circ u$ and $\rho_B = \pi_B \circ u$;

(b) **uniqueness:** this arrow $u\colon X \to P$ is unique.



This is a very wordy definition that takes some getting used to. Let's step through it carefully in the category of sets. We hope that, in **Sets**, the categorical product is the same as the Cartesian product.

Question 3:*Let* A *and* B *be sets. We will prove that* $A \times B = \{(a, b) \mid a \in A, b \times B\}$ *is the product of* A *and* B *in* **Sets**.

(a) *According to the definition of product, we also need functions* $\pi_A\colon A \times B \to A$ *and* $\pi_B\colon A \times B \to B$. *What are these functions? (Hint: the notation is suggestive.)*

(b) *To show that this is a product, we need to demonstrate that it satisfies the definition. Given a set X with functions $\rho_A \colon X \to A$ and $\rho_B \colon X \to B$, we must find a map $u \colon X \to A \times B$ such that $\rho_A = \pi_A \circ u$ and $\rho_B = \pi_B \circ u$. Can you define u?*

(c) *We must also show that u is unique. Given that $\pi_A \circ u = \rho_A$ and $\pi_B \circ u = \rho_B$, conclude that the definition of u from part (b) is the only possible one.*

Hopefully now that you have some idea of how products work, you shouldn't have any trouble finding products in similar categories.

Question 4:

(a) *What is a product object in the category of groups and group homomorphisms?*

(b) *Let X be a topological space with topology τ. Take the open sets of τ as the objects of a category, with an arrow $U \to V$ if and only if $U \subseteq V$. What are product objects in this category?*

Here's one more fun exercise to get used to terminal objects and products.

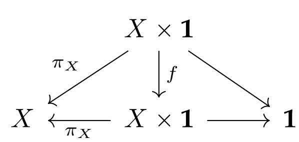
Question 5:*Let $\mathbf{C}$ be a category with terminal object $\mathbf{1}$. Let X be an object of $\mathbf{C}$. We will prove that $X \times \mathbf{1}$ is isomorphic to X.*

(a) *Because $X \times \mathbf{1}$ is the product of X and $\mathbf{1}$, it comes with two arrows. Describe them by stating their domain and codomains.*

(b) *There is another object with arrows to both X and $\mathbf{1}$. What is it? What are the arrows?*

(c) *Use the existence property of products to find a right-inverse u for the arrow $\pi_X : X \times \mathbf{1} \to X$.*

(d) *Now we want to show that u is a left-inverse to π_X as well. To that end, name one arrow $f \colon X \times \mathbf{1} \to X \times \mathbf{1}$ so that the following diagram commutes, meaning that $\pi_X \circ f = \pi_X$.*



(e) *There is another arrow $X \times \mathbf{1} \to X \times \mathbf{1}$ so that the above diagram commutes. What is it? (Hint: it's a composition of two other arrows)*

(f) *Use the uniqueness property of products to conclude that the two arrows from parts (d) and (e) must be equal.*

(g) *Conclude that the arrow u from part (c) is a left-inverse to $\pi_X \colon X \times \mathbf{1} \to X$ as well as a right inverse.*

(h) *Conclude that u is an isomorphism, and $X \cong X \times \mathbf{1}$.*

What is a group?

You probably know what a group is. It's a set G with a binary operation $\cdot$, identity e, and inverses that satisfy some laws.

$$a \cdot e \;=\; a \;=\; e \cdot a \qquad \text{(Identity)}$$
$$a \cdot a^{-1} \;=\; e \;=\; a^{-1} \cdot a \qquad \text{(Inverse)}$$
$$(a \cdot b) \cdot c \;=\; a \cdot (b \cdot c) \qquad \text{(Associativity)}$$

However, everything is a lot more fun with category theory! We can write the definition of a group in any category **C**, so long as there is a terminal object **1** and a product for every pair of objects.

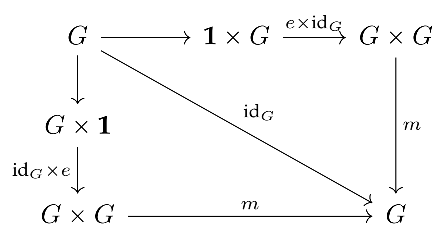
Definition 3. A **group** is an object G in **C** together with three maps:

$$e \colon \mathbf{1} \to G$$
$$i \colon G \to G$$
$$m \colon G \times G \to G.$$

Of these maps, e is supposed to be the identity, i is supposed to be inverse, and m multiplication. In order for our object to be a group, we also need identity, inverse, and associativity laws. In category-land, these laws are interpreted as requiring some diagrams to commute (what else?). The identity and inverse laws appear on the next page; you will draw the diagram for the associativity law in question 9.

Question 6:*Let G be a group. This means that G is a group object in the category of sets. Define the arrows $e \colon \mathbf{1} \to G$, $m \colon G \times G \to G$, and $i \colon G \to G$.*

Here's the identity law in commutative-diagram form.



Question 7:

(a) *Some of the arrows in the diagram for the identity law are unlabelled. What are they?*

(b) *What does the label "*$\mathrm{id}_G \times e$*" on the arrow* $G \times \mathbf{1} \to G \times G$ *mean?*

(c) *If* G *is a group object in the category* **Sets**, *let* $g \in G$. *What is the image of* g *under the three paths around the diagram? Is this the same as the usual identity law?*

This diagram shows the inverse law.

$$\begin{array}{ccccc} G \times G & \xleftarrow{\Delta} & G & \xrightarrow{\Delta} & G \times G \\ \big\downarrow{\scriptstyle \mathrm{id}_G \times i} & & \big\downarrow & & \big\downarrow{\scriptstyle i \times \mathrm{id}_G} \\ & & \mathbf{1} & & \\ & & \big\downarrow{\scriptstyle e} & & \\ G \times G & \xrightarrow{m} & G & \xleftarrow{m} & G \times G \end{array}$$

Question 8:

(a) *What is the arrow labelled* Δ*?*

(b) *What does the label "*$\mathrm{id}_G \times i$*" on the map* $G \times G \to G \times G$ *mean?*

(c) *If* G *is a group object in the category* **Sets**, *let* $g \in G$. *What is the image of* g *under the three paths around the diagram? (Start in the top middle). Is this the same as the usual inverse law?*

Question 9:

(a) *Draw the diagram for the group associativity law. (Hint:* $G \times (G \times G)$ *is isomorphic to* $(G \times G) \times G$*.)*

(b) *If* G *is a group object in* **Sets**, *what is the image of* $(g, h, k) \in G \times G \times G$ *under the two paths around the diagram?*

Some examples

I already revealed that a group object in **Sets** is an ordinary group, but what are group objects in the category of groups? Or maybe the category of rings or vector spaces? We could also talk about group objects in more abstract categories, such as the category of all small categories! (But let's not get carried away here.)

Question 10:*Let* **Top** *be the category of topological spaces and continuous maps between them. What is an example of a group object in* **Top***? Compared to group objects in* **Sets***, what extra condition is imposed on the multiplication and inversion maps?*

Let **Rings** be the category whose objects are rings with identity and arrows are ring homomorphisms preserving **both** the additive and multiplicative identities.

Question 11:

(a) *What is the terminal object* **1** *in this category?*

(b) *Given a ring R that is a group object, describe the arrows e, i and m.*

(c) *Describe all group objects in* **Rings***. (Hint: notice that $e\colon \mathbf{1} \to R$ must preserve both 0 and 1. How are 0 and 1 related in the terminal object of* **Rings***?)*

Now for something a little weird: what is a group object in the category of groups? We'll work this out in the next question.

Question 12: *In the category of groups,* **Groups***, the objects are groups and the arrows are group homomorphisms. This means that $m\colon G \times G \to G$ is a homomorphism.*

(a) *What exactly does it mean for m to be a homomorphism? Given $g = (g_1, g_2)$ and $h = (h_1, h_2)$ elements of $G \times G$, compute $m(gh)$.*

(b) *Show that the image of the identity map $e\colon \{1\} \to G$ is equal the identity of the group G.*

(c) *Using parts (a) and (b), show that $m(g, h) = g \cdot h$ for all $g, h \in G$. Here, $\cdot$ is the group operation in G.*

(d) *Finally, show that $m(g, h) = m(h, g)$ and $g \cdot h = h \cdot g$ for all $g, h \in G$.*

(e) *What are all the group objects in the category of groups?*

We just showed that the group objects in the category of groups are abelian groups! This is called the **Eckmann-Hilton argument**.

Group Homomorphisms

The first thing that you learn in abstract algebra after the definition of a group is the definition of a group homomorphism. We can define these categorically too!

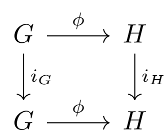
Typically, a group homomorphism is a map $\phi\colon G \to H$ between groups such that

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2).$$

From this, you can conclude that $\phi(e_G) = e_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$. But sadly, to define an abstract group homomorphism between group objects, we need to include these properties in our definition.

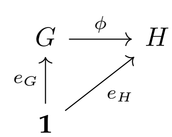
Definition 4. Given two group objects G and H in $\mathbf{C}$, a **group homomorphism** is an arrow $\phi\colon G \to H$ in $\mathbf{C}$ that preserves identity, inverses, and multiplication. Of course, the meaning of "preserves" here is that some diagram commutes!

Saying that ϕ preserves inverses means that this diagram commutes:



Question 13: *Let G be a group object in the category of sets and let $g \in G$. What is the image of g under the two paths around the diagram (start in the top left). Is this the usual statement that homomorphisms preserve inverses?*

Saying that ϕ preserves identity means that this diagram commutes:



Question 14: *Recall that the trivial object in the category of sets is any one-object set $\{*\}$. What is the image of $*$ under the two paths around the diagram? Is this the same as saying that a group homomorphism preserves identity?*

Question 15:

(a) *What diagram guarantees that ϕ preserves multiplication? Draw it!*

(b) *If G is a group object in the category of sets, let $(g_1, g_2) \in G \times G$. What is the image of (g_1, g_2) under the two paths around the diagram?*

Now that we have a mathematical structure of group objects and a notion of arrows between them, we have a category!

Question 16:*Given a category* **C***, show that the collection of group objects in* **C***, together with the group homomorphisms $\phi\colon G \to H$ between group objects, forms a category. That is, show that composition of group homomorphisms is a homomorphism, composition is associative, and there are identity homomorphisms.*

Question 17:*Let* **C** *be the category of group objects in the category of sets. We will show that* **C** *is equivalent to the category* **Groups***.*

(a) *Given an object in* **C***, describe the corresponding object in* **Groups***.*

(b) *Given an object in* **Groups***, describe the corresponding object in* **C***.*

(c) *Conclude that there is a bijection $F\colon \mathbf{C} \to \mathbf{Groups}$ between objects in* **C** *and objects in* **Groups***.*

(d) *Given two objects G, H in* **Groups** *and an arrow $\phi\colon G \to H$ in* **Groups***, interpret ϕ as an abstract group homomorphism.*

(e) *Given two objects G, H in* **C** *and an arrow $\phi\colon G \to H$ in* **C***, show that ϕ is just an ordinary group homomorphism.*

(f) *Conclude that there is a bijection between $\mathrm{Hom}_{\mathbf{C}}(G, H)$ and $\mathrm{Hom}_{\mathbf{Groups}}(F(G), F(H))$, where $\mathrm{Hom}_{\mathbf{D}}(A, B)$ is the set of arrows $A \to B$ in the category* **D***. We also call this bijection F. For any $\phi\colon G \to H$ in* **C***, we have $F(\phi)\colon F(G) \to F(H)$ in* **Groups***.*

(g) *Show that F respects composition. That is, if $G, H, K \in \mathbf{C}$ and $\phi\colon G \to H$ and $\psi\colon H \to K$ are arrows, show that $F(\psi \circ \phi) = F(\psi) \circ F(\phi)$.*

(h) *Conclude that* **C** *and* **Groups** *are equivalent categories.*