LIE ALGEBRAS AND THEIR ROOT SYSTEMS

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In the previous lecture, we classified all possible root systems by transforming them into Dynkin diagrams, and found that there are four infinite families with exactly five exceptions. In this lecture, we will see how Dynkin diagrams correspond bijectively with finite-dimensional simple complex Lie algebras, and therefore the classification of Dynkin diagrams is actually a classification of all such Lie algebras. As for why we should want to classify all such Lie algebras, there are plenty of applications of Lie algebras to other fields of math and even to physics that make them very interesting objects of study. A few reasons why Lie algebras are useful and interesting are listed below.

- There are many physics applications, especially in quantum physics. For example, the standard model of particle physics is encapsulated in the representations of the Lie group $SU(3) \times SU(2) \times U(1)$, and we can study these representations by studying representations of the associated Lie algebras $\mathfrak{su}(3)$, $\mathfrak{su}(2)$, and $\mathfrak{u}(1)$. For more about physics applications, see [Ram10].
- The group of an elliptic curve E over $\mathbb{C}$ is isomorphic to a torus, $\mathbb{C}/\Lambda$, which is a complex Lie group. The Lie algebra associated to this Lie group is related to the differentials of the curve, which are in turn useful in the study of E.
- A Lie group G has a Lie algebra $\mathfrak{g}$ associated to it, which is defined as the tangent space to G at the identity. By studying the Lie algebra, we are able to work with all the tools of linear algebra to study the group.

It should be mentioned that Lie algebras and Lie groups, although closely related to geometry, are not considered part of geometric group theory, which has a more discrete flavor to it. but it is a nice application of root systems and Dynkin diagrams, which are closely related to geometric group theory.

So what is a Lie algebra?

Definition 1. *A Lie algebra $\mathfrak{L}$ is a vector space with a skew-symmetric bilinear map, called the Lie bracket or commutator and written as $[\,\cdot\,,\,\cdot\,]\colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$, which satisfies the Jacobi identity:*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \forall x, y, z \in \mathfrak{L}.$$

Note that the Lie bracket of a Lie algebra need not be associative, and in fact, it will likely not be. In this case, we call the algebra *abelian*.

Example 2. *Some examples of Lie algebras:*

- **2.1.** *Any vector space V is a Lie algebra, with bracket $[u,v] = 0$ for all $u, v \in V$. This type of algebra is called* abelian.
- **2.2.** *For a field k of characteristic zero, we have the classical matrix algebras $\mathfrak{gl}_n(k)$ of $n \times n$ matrices over k, $\mathfrak{sl}_n(k)$ the subalgebra of $\mathfrak{gl}_n(k)$ of those $n \times n$ matrices with determinant one. There are also the algebras $\mathfrak{so}_n(\mathbb{R})$ of $n \times n$ orthogonal real matrices, or $\mathfrak{su}_n(\mathbb{C})$ of $n \times n$ unitary complex matrices. For each of these, the commutator is given by $[X,Y] = XY - YX$.*
- **2.3.** *For any Lie group G, the tangent space at the identity is a Lie algebra, usually denoted $\mathfrak{g}$.*

2.4. *For any associative algebra A, we may define a Lie algebra $\mathfrak{L}(A)$ over A by defining the Lie bracket as $[x, y] = xy - yx$.*

There are a few more definitions needed for this lecture.

Definition 3. *An ideal I of a Lie algebra $\mathfrak{L}$ is a vector subspace of $\mathfrak{L}$ such that $[i, x] \in I$ for all $i \in I$ and $x \in \mathfrak{L}$. This is a two sided ideal by the skew-symmetric nature of the commutator. If a Lie algebra has no nontrivial ideals, it is called simple.*

Finding the root system of a Lie algebra

We focus on finding a root system for a Lie algebra, since we understand the correspondence between root systems and Dynkin diagrams, which was described in the previous lecture.

Let $\mathfrak{L}$ be a complex simple Lie algebra with vector-space basis $\{x_1, \ldots, x_n\}$. To know the Lie algebra structure for $\mathfrak{L}$ given the basis, we need to know the *structure constants*: numbers f_{ijk} such that

$$[x_i, x_j] = \sum_{k=1}^{n} f_{ijk} x_k \ .$$

The information of these structure constants is actually encoded entirely in the root system of the Lie algebra, as we will see shortly. This will be much easier if as many of the structure constants as possible are zero, so we will find a new basis for $\mathfrak{L}$. In this basis, we want $[x_i, x_j] = 0$ for as many i, j as possible, so we look for a Cartan subalgebra.

Definition 4. *A Cartan subalgebra $\mathfrak{h}$ for a Lie algebra $\mathfrak{L}$ is an abelian, diagonalizable subalgebra which is maximal under set inclusion. The dimension of $\mathfrak{h}$ is the rank of $\mathfrak{L}$.*

It won't matter which Cartan subalgebra we choose, because they will all be conjugate under automorphisms of the Lie algebra, and so they have the same dimension. Thus, the rank is independent of the choice of $\mathfrak{h}$, and so an invariant of $\mathfrak{L}$. Furthermore such an algebra will always exist, at least for finite dimensional Lie algebras over $\mathbb{C}$.

So let's pick a Cartan subalgebra $\mathfrak{h}$ for our Lie algebra $\mathfrak{L}$, and define a basis $\{H_1, \ldots, H_r\}$ for $\mathfrak{h}$. Because $\mathfrak{h}$ is abelian, $[H_i, H_j] = 0$ for all i, j. We will extend this basis for $\mathfrak{h}$ to a basis of $\mathfrak{L}$, and thereby obtain a basis which has much simpler and more convenient commutator relations. We can analyze the commutator relations by looking at the linear operators $[H_i, \cdot\,]$. These are called the adjoint operators of the H_i, denoted adj_{H_i}. The adjoint operators form a representation of $\mathfrak{L}$ called the adjoint representation.

Fact 5. *Pairwise commuting, diagonalizable linear operators share a common set of eigenvectors.*

Fact 6. *If $H_1, H_2 \in \mathfrak{h}$, then the linear operators adj_{H_1} and adj_{H_2} also commute, and are diagonalizable. So they share a common set of eigenvectors.*

Proof. We only show that they commute. Using the Jacobi identity,

$$[H_1, [H_2, X]] = -[H_2, [X, H_1]] - [X, [H_1, H_2]] = -[H_2, [X, H_1]] - [X, 0] = [H_2, [H_1, X]] + 0 \quad \square$$

By this fact, these operators adj_{H_i} have a set of common eigenvectors, and moreover, by the spectral theorem we have a decomposition of $\mathfrak{L}$ into shared eigenspaces g_α of the adjoint operators, as

$$\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \ ,$$

where the labels $\alpha \in \Phi \subseteq \mathbb{R}^r$ are the eigenvalues of adj_{H_i} on the eigenspace $\mathfrak{g}_\alpha$, in particular, α_i is the eigenvalue for adj_{H_i} on g_α. So for each $E \in \mathfrak{g}_\alpha$, $[H_i, E] = \alpha_i E$. These α are called, suggestively, the *roots* of $\mathfrak{L}$. It is also often helpful to think of the root α as belonging to the dual space $\mathfrak{h}^*$, with $\mathrm{adj}_{H_i} X = \alpha(H_i) X$.

Theorem 7. *The set of roots Φ forms a root system in $\mathbb{R}^r$. Furthermore, each eigenspace $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ is one-dimensional.*

Suppose that $\mathfrak{g}_\alpha$ is the span of E_α for each $\alpha \in \Phi$. Then we may extend the basis $\{H_1, \dots, H_r\}$ for $\mathfrak{h}$ into a basis $\{H_1, \dots, H_r\} \cup \{E_\alpha \colon \alpha \in \Phi\}$ for $\mathfrak{L}$, that satisfies the commutator relations $[H_i, H_j] = 0$ and $[H_i, E_\alpha] = \alpha_i E_\alpha$. So it remains to figure out what $[E_\alpha, E_\beta]$ is for $\alpha, \beta \in \Phi$. Let's first consider the case where $\beta = -\alpha$. First, observe that for any H_i,

$$\begin{aligned}[H_i, [E_\alpha, E_{-\alpha}]] &= -[E_\alpha, [E_{-\alpha}, H_i]] - [E + -\alpha, [H_i, E_\alpha]] \\ &= [E_\alpha, -\alpha_i E_{-\alpha}] - [-E_\alpha, \alpha_i E_\alpha] \\ &= -\alpha_i [E_\alpha, E_{-\alpha}] + \alpha_i [E_\alpha, E_{-\alpha}] = 0\end{aligned}$$

So $[E_\alpha, E_{-\alpha}]$ commutes with each of the H_i, and therefore must be in $\mathfrak{h}$ by maximality of the Cartan subalgebra. Hence, we can conclude that $[E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \beta_i H_i$ for some β_i. By considering what is called the Killing form of $\mathfrak{L}$, which essentially defines an inner product on the adjoint operators, we may show the fact below, but we will not prove it here.

Fact 8. *The coefficients of $[E_\alpha, E_{-\alpha}]$ as an element of $\mathfrak{h}$ are given by the root vector α. In particular,*

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha_i H_i \ .$$

So the only structure constants we don't know are those for $[E_\alpha, E_\beta]$ for $\beta \neq -\alpha$. Once again, we can heuristically determine what it may look like by considering the action of the adjoint representation on it:

$$[H_i, [E_\alpha, E_\beta]] = -[E_\beta, [H_i, E_\alpha]] - [E_\alpha, [E_\beta, H_i]] = -[E_\beta, \alpha_i E_\alpha] - [E_\alpha, -\beta_i E_\beta] = (\alpha_i + \beta_i)[E_\alpha, E_\beta]$$

This shows that $[E_\alpha, E_\beta]$ is an eigenvector of adj_{H_i}, and so it should be proportional to $E_{\alpha+\beta}$, if $\alpha + \beta$ is a root. The exact constant can be found explicitly again with the Killing form, but again, we will only state it.

Fact 9. *For the basis $\{H_1, \dots, H_r\} \cup \{E_\alpha \colon \alpha \in \Phi\}$ of $\mathfrak{L}$, the structure constants are*

$$[H_i, H_j] = 0 \qquad [H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha_i H_i, \qquad [E_\alpha, E_\beta] = \begin{cases} \frac{4(\alpha,\beta)}{|\alpha|^2|\beta|^2} E_{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & \alpha + \beta \notin \Phi \end{cases}$$

Recovering a Lie algebra from its root system

So how do we recover the Lie algebra given a root system Φ? We use the commutator relations from fact 9. Let Δ be the set of simple roots for Φ. For each $i \in \{1, \dots, |\Delta|\}$, we have a basis element H_i for our Cartan subalgebra, and these satisfy $[H_i, H_j] = 0$. For each other $\alpha \in \Phi$, we have E_α such that $[H_i, E_\alpha] = \alpha_i E_\alpha$ for all i, and $[E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha_i H_i$, and finally, we demand that $[E_\alpha, E_\beta]$ satisfies the relation in fact 9 for all $\alpha, \beta \in \Phi$. This defines a basis for the Lie algebra, and gives us the structure constants as well, so we can determine all of the commutator relations between any elements. This shows how to retrieve a Lie algebra from its root system.

References

[CH14] Hael Collins and Richard Holman. Group theory with physics applications. University Lecture, 2014.

[EW06] Karin Erdmann and Mark J Wildon. *Introduction to Lie algebras*. Springer, 2006.

[FH91] William Fulton and Joe Harris. *Representation theory: a first course*, volume 129. Springer, 1991.

[Ram10] Pierre Ramond. *Group theory: a physicist's survey*. Cambridge University Press, 2010.

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