ROOT SYSTEMS AND DYNKIN DIAGRAMS

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In 1969, Murray Gell-Mann won the nobel prize in physics "for his contributions and discoveries concerning the classification of elementary particles and their interactions." He is the scientist who first used the word quark, and he was the first to describe the $SU_3(\mathbb{C})$ flavor symmetry of the hadrons. By organizing the particles according to the *root system* of the lie algebra $\mathfrak{su}_3(\mathbb{C})$ associated to the group of symmetries, he predicted the existence of new particles, which were later found experimentally. So root systems provide a key element in mathematical models for quantum systems, which is one reason that we might want to understand them.

Root systems are also the key ingredient in the classification of finite-dimensional, simple Lie algebras. The symmetries of root systems are the Weyl groups, one of the types of Coxeter groups, which are of interest in geometry group theory.

ROOT SYSTEMS

Let (α, β) be the standard Euclidean inner product on \mathbb{R}^n .

Definition 1. A root system Φ is a set of vectors in \mathbb{R}^n such that:

- (1) Φ spans \mathbb{R}^n and $0 \notin \Phi$.
- (2) If $\alpha \in \Phi$ and $\lambda \alpha \in \Phi$, then $\lambda = \pm 1$.
- (3) If $\alpha \in \Phi$ is closed under reflection through the hyperplane normal to α .
- (4) If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$

The elements $\alpha \in \Phi$ are called *roots*. An immediate consequence of (4) is that there are only finitely many allowed angles between vectors of a root system.

Fact 2. If θ is the angle between α and β , $\theta \in \{0, \pi, \pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}$.

Proof. Observe that

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{4(\alpha, \beta)^2}{|\alpha|^2 |\beta|^2} = 4\cos^2 \theta_{\alpha\beta} ,$$

and the left hand side is an integer, while the right hand side is between 0 and 4, which necessarily restricts the possible values of θ . Note that the only scenario for which $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4$ is when $\beta = \pm \alpha$.

From this we can see that the ratio between lengths of vectors in a root system is fixed by the angle between them. This also shows that root systems must be finite sets. Some examples of root systems follow:

Example 3. This is the root system A_2 , depicted as vectors in \mathbb{R}^2 .



Example 4. The set of standard basis vectors and their opposites, $\{\pm e_i \mid 1 \leq i \leq n\}$, is a root system. It is a boring example of a root system, because reflection through any one of the hyperplanes normal to a vector leaves all others fixed.

Example 5. The root system $\{\pm(e_i - e_j) \mid 1 \le i < j \le n+1\}$ is an important example from Lie algebras. It is the root system of the Lie algebra $A_n = \mathfrak{sl}_{n+1}(\mathbb{C})$.

The symmetry of a root system defined by "reflect through the hyperplane perpendicular to α " is given by $\sigma_{\alpha}(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$, and the group generated by $\{\sigma_{\alpha} \mid \alpha \in \Phi\}$ is the Weyl group of the system. Example (5) is particularly boring when considering its Weyl group, and that's because it can be broken down into many smaller root systems: it is what we call decomposable.

Definition 6. A root system Φ is decomposable if it can be written as a disjoint union $\Phi = \Phi_1 \cup \Phi_2$ such that $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in \Phi_1$, $\alpha_2 \in \Phi_2$. If Φ is not decomposable it is indecomposable.

Every root system Φ can be written as the disjoint union of indecomposable root systems. We may obtain such a decomposition by defining an equivalence relation \sim on Φ , with $\alpha \sim \beta$ if and only if there is a sequence $\alpha = \gamma_1, \gamma_2, \ldots, \gamma_s = \beta$ such that $(\gamma_i, \gamma_{i+1}) \neq 0$ for all *i*. The equivalence classes under this relation are the indecomposable root systems that compose Φ .

To specify a root system, we don't need to specify all of the roots. We may recover it from a smaller set of roots called the simple roots.

Definition 7. A set of positive roots for a root system is a set $\Phi^+ \subseteq \Phi$ such such that

(1) For all $\alpha \in \Phi$, exactly one of α and $-\alpha$ is in Φ^+ .

(2) If $\alpha, \beta \in \Phi^+$, if $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi^+$.

The non-positive roots are called the negative roots.

Definition 8. $\alpha \in \Phi^+$ is a simple root for Φ if it is not the sum of two other positive roots. The set of simple roots is Δ .

At first glance, there is no one favored set of simple roots or positive roots. In fact, there are many different choices of simple and positive roots, but they are all conjugate by an element of the Weyl group of the root system. If we want to find a set of simple roots, we must first determine when two roots may be added together.

Fact 9. If two roots $\alpha, \beta \in \Phi$ are such that the angle θ between them is strictly obtuse, then $\alpha + \beta \in \Phi$. If θ is strictly acute and $|\beta| \ge |\alpha|$, then $\alpha - \beta \in \Phi$.

Proof Sketch. From before, we know that there are only so many allowed angles between roots. By casing on possible values for $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ given that their product is an integer between 0 and 3, we can determine when $\alpha + \beta$ or $\alpha - \beta$ is a root.

In particular, this tells us that the angles between simple roots are necessarily obtuse. We also want to know how many simple roots are in our root system. This is answered by considering a harder question: is Δ a basis for \mathbb{R}^n ? Of course it is, because otherwise this would be boring.

Theorem 10. Every root system has a set of simple root such that for each $\alpha \in \Phi$ may be written as

$$\alpha = \sum_{\delta \in \Delta} k_{\delta} \delta,$$

with $k_{\delta} \in \mathbb{Z}$, and each k_{δ} has the same sign.

Proof. The idea of this proof is to draw a hyperplane through the origin that doesn't contain any vectors in it, and choose the simple roots as the ones that are closest to the hyperplane.



So we choose a hyperplane that doesn't contain any roots, and let z be any vector in this hyperplane. Define $\Phi^+ = \{\alpha \in \Phi \mid (\alpha, z) > 0\}$ and Δ to be the set of positive roots which are not the sum of any two elements in Φ^+ . Clearly Φ^+ satisfies the properties of a set of positive roots. So it remains to show that Δ spans Φ . Since for each root β , either $\beta \in \Phi^+$ or $-\beta \in \Phi^+$, it suffices to show this for $\beta \in \Phi^+$. By contradiction.

Let β be an element of Φ^+ that is not a linear combination of elements of Δ as above. Further assume that (β, z) is minimal among such choices of β . Since $\beta \notin \Delta$, then β can be written as the sum of two positive roots, say $\beta = \alpha + \gamma$. Then $0 < (\beta, z) = (\gamma, z) + (\alpha, z)$, but since both α and γ are positive, then both $(\alpha, z) > 0$ and $(\gamma, z) > 0$, and therefore $(\alpha, z) < (\beta, z)$ and $(\gamma, z) < (\beta, z)$. If both α, γ can be written as such a linear combination, then so can $\beta = \alpha + \gamma$, so we are done. So at least one of α, γ is NOT a linear combination of elements of Δ with all positive coefficients, say γ . But then $(\gamma, z) < (\beta, z)$, so this contradicts the minimality condition on β .

From this, we see that Δ spans Φ and therefore Δ spans \mathbb{R}^n . To show that it's a basis, we must show that Δ is independent.

Lemma 11. Δ is a set of independent vectors, and therefore a basis for \mathbb{R}^n .

Proof. Suppose for the sake of contradiction that $\sum_{\delta \in \Delta} k_{\delta} \delta = 0$ for some coefficients k_{δ} , k_{δ} not all zero. Let

$$x = \sum_{\substack{\delta \in \Delta \\ k_{\delta} > 0}} k_{\delta} \delta$$
 and $y = \sum_{\substack{\delta \in \Delta \\ -k_{\delta} < 0}} k_{\delta} \delta.$

Note that x - y = 0, so x = y. Then

$$|x|^{2} = (x, x) = (x, y) = \sum_{\substack{\delta \in \Delta \\ k_{\delta} > 0}} \sum_{\substack{\gamma \in \Delta \\ k_{\gamma} < 0}} -k_{\delta} k_{\gamma}(\gamma, \delta).$$

Since the angles between any two simple roots must be obtuse, then $(\gamma, \delta) < 0$ for all pairs γ, δ . So this is the sum of strictly negative numbers, but $|x|^2$ must be positive. Thus we obtain a contradiction, and so there are no nontrivial linear combinations.

It's worth briefly mentioning how to recover a root system from the simple roots. Using the closure of Φ under reflections σ_{α} (elements of the Weyl group), we can reconstruct the entire root system. In fact, if W is the Weyl group of Φ , then W is generated by $\{\sigma_{\delta} \mid \delta \in \Delta\}$. This is illustrated on the next page.

Example 12. The root system G_2 has simple roots given by (1,0) and $(-3/2,\sqrt{3}/2)$, pictured as the thick lines below. By reflecting about the lines through the origin perpendicular to them, we obtain the entire root system G_2 .



DYNKIN DIAGRAMS

We can condense the information that determines a root system even further by drawing the Dynkin diagram for the root system. For example, the Dynkin diagram for example 12 is \bigcirc , which significantly condenses the information in a root system. The last time we drew the root system G_2 , it took up half a page, but this diagram barely takes any more room than the word "diagram"! Dynkin diagrams are also the tool by which we classify the possible root systems. Why do we want to classify the possible root systems? Well, root systems correspond bijectively to finite dimensional, simple Lie algebras, and so by classifying the root systems we also classify all finite dimensional, simple Lie algebras.

To build a Dynkin diagram, we take the set of simple roots Δ and draw one node for each simple root. This means that the number of vertices for a diagram is the same as the rank of the root system, which is the dimension of the ambient space. For every pair of simple roots α and β , we draw a number of lines between their vertices equal to $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$, which was guaranteed to be an integer between zero and three by (2). If one simple root is longer than another, we draw an arrow pointing to the shorter one (you may also want to think about this as a "greater than" sign showing which is longer). If two simple roots have the same length, we omit the arrow.

Example 13. The root system A_2 from example (3) has two simple roots, given by $\alpha = (0, 1)$ and $\beta = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$. The angle between these is $5\pi/2$ and so we draw a single line between them. The diagram for this root system is: \bigcirc .

We already saw that from a set of simple roots, we may recover the entire root system. And the Dynkin diagram gives us the angles and relative lengths of the simple roots, which allows us to recover the set of simple roots up to scalar multiples and rotation in the plane. So now we will move on to the classification of the root systems. There are four lemmas between us and our goal. The first will show that connected Dynkin diagrams must be trees.

Lemma 14. Ignoring multiplicity of edges, a connected Dynkin diagram with n nodes has n - 1 edges.

Proof. Let $\Phi \subseteq \mathbb{R}^n$ be an irreducible root system and let Δ be the set of simple roots. Since Δ is a basis, there are *n* simple roots, and so *n* vertices in the diagram. Let *e* be the number of edges, not counting multiplicity. Now define $\Delta' = \{\frac{\delta}{|\delta|} \mid \delta \in \Delta\}$, and enumerate this basis as v_1, \ldots, v_n . Let $v = \sum_{i=1}^n v_i$. Then

$$0 < |v|^{2} = (v, v) = \sum_{i=1}^{n} |v_{i}| + 2\sum_{i < j} (v_{i}, v_{j}) = n + 2\sum_{i < j} (v_{i}, v_{j}).$$

Since the angle between any two simple roots is obtuse, then $(v_i, v_j) < 0$ for all $i \neq j$. Then we observe that

$$n > -\sum_{i < j} 2(v_i, v_j) = \sum_{i < j} \sqrt{\langle v_i, v_j \rangle \langle v_j, v_i \rangle} \ge e.$$

Since n > e, then $e \le n-1$. But since the diagram is connected, we must have that e = n-1. \Box

Proof. We only consider connected diagrams, The angles between three linearly independent vectors must add up to less than 2π . This rules out all other possibilities when we add the angles determined by the diagram.

Lemma 16 (Shrinking Lemma). If $\cdots \bigcirc \bigcirc \cdots$ is a valid Dynkin diagram (i.e. represents a valid set of simple roots), then we may shrink along the single edge to get another valid Dynkin

diagram $\cdots \bigcirc \cdots$. Similarly, if $\cdots \bigcirc \cdots$ is part of a valid diagram, then we may collapse the two vertices into a single vertex with a double line, as such $\cdots \bigcirc \bigcirc$.

Proof. If α and β are the simple roots corresponding to the vertices to be collapsed, then we simply check that the set Δ' formed by $\Delta \setminus \{\alpha, \beta\} \cup \{\alpha + \beta\}$ is a valid set of simple roots of one less rank, and therefore determines a valid diagram.

Combining lemma 15 and lemma 16, we can make a few observations about which diagrams are allowed. In particular, we see by shrinking diagrams to invalid three vertex diagrams, we may rule out some larger diagrams. The first observation should be that the only valid diagram with three lines is G_2 , as in example 12. Moreover, diagrams cannot have more than one double line or more than one branch, and cannot have both a branch and double line. So we are left with only a few possible diagrams, and must rule out the remaining ones by hand:

Lemma 17. The following Diagrams are invalid. If there is a double edge, both diagrams, either with an arrow pointing left or with an arrow pointing right, are invalid.



Proof. These are all invalid diagrams because the set of simple roots they define are not independent vectors. We only show this for the diagram with labelled nodes, assuming the arrow points to the left. So α_4, α_5 are longer than the others by $\sqrt{2}$. The claim is that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 = 0$. It suffices to check that $|\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5|^2 = 0$. Note that α_i and α_j are orthogonal when |i - j| > 1 because there are no lines in the diagram between them. So

$$\begin{aligned} |\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5|^2 \\ &= |\alpha_1|^2 + 4|\alpha_2|^2 + 9|\alpha_3|^2 + 4|\alpha_4|^2 + |\alpha_5|^2 + 4(\alpha_1, \alpha_2) + 12(\alpha_2, \alpha_3) + 12(\alpha_3, \alpha_4) + 4(\alpha_4, \alpha_5) \\ &= 1 + 4 + 9 + 8 + 2 + 4\left(\frac{-1}{2}\right) + 12\left(\frac{-1}{2}\right) + 12\left(\frac{-1}{\sqrt{2}}\right)\left(\sqrt{2}\right) + 4\left(\frac{-\sqrt{2}}{2}\right)\left(\sqrt{2}\right) \\ &= 24 - 2 - 6 - 12 - 4 \\ &= 0 \end{aligned}$$

Combining these four lemmas leaves us with only so many possible valid diagrams, as below.

Theorem 18 (Classification Theorem). The possible connected Dynkin diagrams are either one of the four infinite families



or one of exactly five exceptional diagrams. Since indecomposable root systems correspond to connected Dynkin diagrams bijectively, this also classifies all possible irreducible root systems.



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