Complex Manifolds

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Remark 1 (Prerequisites)**.**

(1) Differential Geometry: Manifolds, differential forms, de Rahm cohomology, metrics, vector bundles.

(2) Complex analysis in one variable, and a willingness to accept that these statements generalize to several variables.

(3) Some elementary algebraic geometry, sheaf theory.

With that out of the way, let's move on to some mathematics.

Definition 2. A **complex manifold** of dimension n is a Hausdorff topological space M equipped with an open cover $\{U_i \mid i \in I\}$ and maps $\phi_i \colon U_i \to \mathbb{C}^n$ which give a homeomorphism of U_i with an open subset of $\mathbb{C}^n$, and the transition maps

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are holomorphic.

This looks a lot like the definition of a real manifold, but the key point here is that we insist that the transition maps are not just differentiable, but instead holomorphic (sometimes called complex analytic). Let's see some examples.

Example 3. (1) $M = \mathbb{C}^n$.

(2) If you like compact spaces (which most people do), let $\Lambda \subseteq \mathbb{C}^n$ be a **rank** $2n$ **lattice**, i.e. Λ is an additive subgroup of $\mathbb{C}^n$ isomorphic to $\mathbb{Z}^{2n}$ that spans $\mathbb{C}^n$ as a $\mathbb{R}$-vector space. (The reason for the spanning condition is to disallow such silliness as $\mathbb{Z} + \mathbb{Z}\sqrt{2} \subseteq \mathbb{C}$). Then take $M = \mathbb{C}^n/\Lambda$. This is called a **complex torus**. This seems pretty average but it's a huge field of math. When $n = 1$ this is elliptic curves.

(3) n-dimensional projective space $\mathbb{CP}^n$ (which we often write as $\mathbb{P}^n$).

$$\mathbb{P}^n := \left(\mathbb{C}^n \backslash \{0\}\right) / \mathbb{C}^\times$$

where $\lambda \in \mathbb{C}^\times$ acts by $\lambda \cdot (x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$. So far we've defined a topological space, but we need charts and transition maps for this to be a complex manifold. There's a nice natural open cover: let $U \subseteq \mathbb{P}^n$ be the open subset

$$U_i := \{(x_0, \ldots, x_n) \mid x_i \neq 0\}.$$

A point $p \in U_i$ can be written uniquely as

$$p = (x_0, \ldots, 1, \ldots, x_n)$$

with $x_i = 1$. This gives a map

$$\begin{array}{rccc} \phi_i \colon & U_i & \longrightarrow & \mathbb{C}^n \\ & (x_0, \ldots, x_n) & \longmapsto & \left({}^{x_0}\!/_{x_1}, \ldots, \underset{\substack{\uparrow \\ i}}{\widehat{{}^{x_i}\!/_{x_i}}}, \ldots, {}^{x_n}\!/_{x_i}\right) \end{array}$$

Then $\phi_i \circ \phi_j^{-1}$ is the map

$$(y_0, \ldots, \widehat{y_j}, \ldots, y_n) \xmapsto{\phi_j^{-1}} (y_0, \ldots, \underset{\substack{\uparrow \\ j}}{1}, \ldots, y_n) \xmapsto{\phi_i} \left(\frac{y_0}{y_1}, \ldots, \underset{\substack{\uparrow \\ i}}{\widehat{\frac{x_i}{x_i}}}, \ldots, \underset{\substack{\uparrow \\ j}}{\frac{1}{y_i}}, \ldots, \frac{y_n}{y_i} \right)$$

and therefore is holomorphic. Note that $\mathbb{P}^n$ is compact.

(4) For a special case, $\mathbb{P}^1 = U_1 \cup \{(1,0)\}$. We can write $U_1 = \{(x,1) \mid x \in \mathbb{C}\}$, so $U_1 \simeq \mathbb{C}$, and write the extra point $(1,0)$ as ∞. This is the **Riemann sphere** $\mathbb{C}_\infty$ with a map $\mathbb{P}^1 \to \mathbb{C}_\infty$ given by $(x,y) \mapsto {}^x/_y$.

(5) Let $f_1, \ldots, f_m \in \mathbb{C}[z_0, \ldots, z_n]$ be **homogeneous** polynomials, i.e. each monomial in f_i is of the same degree. Hence, $f_i(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f_i(z_0, \ldots, z_n)$, where $d = \deg f_i$. Define

$$Z(f_1, \ldots, f_m) = \{(z_0, \ldots, z_n) \in \mathbb{P}^n \mid f_i(z_0, \ldots, z_n) = 0 \;\forall i\}$$

Suppose $M = Z(f_1, \ldots, f_m)$ is of pure dimension d (ruling out such silly examples as the union of a plane and a line), and furthermore the rank of the Jacobian matrix is at least $n - d$,

$$\operatorname{rank} \begin{bmatrix} \partial f_1/\partial z_0 & \cdots & \partial f_1/\partial z_n \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial z_0 & \cdots & \partial f_m/\partial z_n \end{bmatrix} \geqslant n - d$$

at every point of M, then M is a complex manifold. (This is a version of the implicit function theorem).

Proof of claim. Let $p \in M$, say $p \in U_0$ without loss of generality, so we can set $z_0 = 1$. After reordering the $z_i's$ and the $f_j's$, we can assume that the upper left-hand $(n-d) \times (n-d)$ submatrix in the corner has rank $n - d$. We can then construct a map

$$U_0 = \mathbb{C}^n \xrightarrow{(f_1, \ldots, f_{n-d})} \mathbb{C}^{n-d}.$$

Then the implicit function theorem implies that the equations

$$f_1(z_1, \ldots, z_n) = \ldots = f_{n-d}(z_1, \ldots, z_n) = 0$$

can be solved for $z_1, \ldots, z_{n-d}$ in terms of $z_{n-d+1}, \ldots, z_n$ locally near p in a holomorphic manner. This gives a parametrization of an open neighborhood of p by $z_{n-d+1}, \ldots, z_n$. Such a solution gives also a solution to $f_1 = \ldots = f_m = 0$, as otherwise the dimension would be less than d, but we assumed that M is of pure dimension d. □

(6) For a concrete example of the previous construction, consider the zero set of $f = z_1^2 z_0 - (z_2^3 + z_2 z_0^2)$. Write $E = Z(f) \subseteq \mathbb{P}^2$. The matrix

$$\left[\partial f/\partial z_0, \quad \partial f/\partial z_1, \quad \partial f/\partial z_2 \right] = \left[z_1^2 - 2z_0 z_2, \quad 2z_0 z_1, \quad 3z_2^2 + z_0^2 \right]$$

has rank 1 if it's nonzero. If all derivatives of f are zero, then either $z_0 = 0$ or $z_1 = 0$ because $0 = {}^{\partial f}/_{\partial z_1} = 2z_0 z_1$. If $z_0 = 0$, it must be that $z_2 = z_1 = 0$, which doesn't happen. If $z_1 = 0$, $z_0 z_2 = 0$ so $z_0 = z_2 = 0$ which doesn't happen. E is a 1-dimensional complex submanifold, isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$.

You may be thinking that I'm using this as a trick to just do algebraic geometry, but it's really an illustration of the following important theorem. We won't prove this theorem in the course, but it's useful to know.

Theorem 4 (Chow's Theorem)**.** A compact complex submanifold of $\mathbb{P}^n$ is the of this form: $Z(f_1, \ldots, f_m)$.

So in some sense, complex geometry is really closely related to algebraic geometry. But on the other hand, there are some complex manifolds that cannot be embedded in projective space, so studying complex manifolds is worthwhile on it's own too.

Example 5. This is an example of a complex manifold that cannot be embedded in projective space. The **Hopf surface** is

$$M = (\mathbb{C}^n \backslash \{0\}) \,/ \sim$$

where $(z_0, \ldots, z_n) \sim (2z_1, \ldots, 2z_n)$. M is homeomorphic to $S^1 \times S^{2n-1}$ by

$$\begin{array}{ccc} M & \longrightarrow & S^1 \times S^{2n-1} = (\mathbb{R}/\mathbb{Z}) \times S^{2n-1} \subseteq \mathbb{C}^n \\ z & \longrightarrow & \left(\log_2 \|z\| \mod \mathbb{Z},\ {}^{z}/_{\|z\|}\right) \end{array}$$

This cannot be embedded in projective space for $n > 1$.

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Definition 6. If M is a complex manifold, $U \subseteq M$ open, then a function $f\colon U \to \mathbb{C}$ is **holomorphic** if, for each coordinate chart $\phi_i\colon U_i \to \mathbb{C}^n$, $f \circ \phi_i^{-1}\colon \phi_i(U \cap U_i) \to \mathbb{C}^n$ is holomorphic.

A continuous map $f\colon M \to N$ between two complex manifolds is **holomorphic** if for every open set $U \subseteq N$ and $g\colon U \to \mathbb{C}$ holomorphic, $g \circ f\colon f^{-1}(U) \to \mathbb{C}$ is also holomorphic.

We should think of this definition by saying that the interesting data on a complex manifold are the functions, and that this structure is preserved by maps between manifolds.

Locally, if $U \subseteq \mathbb{C}^n$, $V \subseteq \mathbb{C}^m$ are open, a holomorphic map $f\colon U \to V$ is given by $f = (f_1, \ldots, f_m)$ with $f_i\colon U \to \mathbb{C}$ holomorphic.

A recurring theme in this course will be that complex geometry is more rigid than differential geometry, as in the following example.

Example 7. Let $f\colon \mathbb{C}^n/\Lambda_1 \to \mathbb{C}^n/\Lambda_2$ be a holomorphic map between complex tori. Assume $f(0) = 0$.

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\widehat{f}} & \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{C}^n/\Lambda_1 & \xrightarrow{f} & \mathbb{C}^n/\Lambda_2 \end{array}$$

The map $\widehat{f}$ is a lift of f with $\widehat{f}(0) = 0$. So maps between tori are the same as maps $\mathbb{C}^n \to \mathbb{C}^n$, with some restrictions. So long as the map $\widehat{f}\colon \mathbb{C}^n \to \mathbb{C}^n$ satisfies, for all $\lambda_1 \in \Lambda_1$,

$$\widehat{f}(z + \lambda_1) = \widehat{f}(z) + \lambda_2 \tag{1}$$

for some $\lambda_2 \in \Lambda_2$. By continuity of $\widehat{f}$, λ_2 is independent of z, and taking $z = 0$ we see that $\widehat{f}|_{\Lambda_1}\colon \Lambda_1 \to \Lambda_2$ is well-defined.

If we differentiate (1), we get

$$\frac{\partial \widehat{f}(z + \lambda_1)}{\partial z_i} = \frac{\partial \widehat{f}(z)}{\partial z_i}$$

so ${}^{\partial \widehat{f}}/_{\partial z_i}$ is periodic, holomorphic, and hence bounded and therefore by Liouville's theorem, ${}^{\partial \widehat{f}}/_{\partial z_i}$ is constant.

Thus, $\widehat{f}$ is linear. So a holomorphic map $f\colon \mathbb{C}^n/\Lambda_1 \to \mathbb{C}^n/\Lambda_2$ is induced by a linear map $\widehat{f}\colon \mathbb{C}^n \to \mathbb{C}^n$ with $\widehat{f}(\Lambda_1) = \Lambda_2$.

Example 8. If $n = 1$, then a lattice is given by $\mathbb{C}/\langle \tau_1, \tau_2 \rangle$ with $\tau_1, \tau_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. Alternatively, dividing by τ_1, a lattice is of the form $\mathbb{C}/\langle 1, \tau \rangle$ for $\tau = {}^{\tau_2}/_{\tau_1}$. Note that $\langle 1, \tau \rangle = \langle 1, -\tau \rangle$, so we may assume τ has positive imaginary part. We express this by saying that τ lies in the upper half-plane $\mathcal{H}$.

Definition 9. The **upper half plane** $\mathcal{H}$ is defined by $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$

Exercise 10. Show if $\tau, \tau' \in \mathcal{H}$, there is an isomorphism $f\colon \mathbb{C}/\langle 1, \tau \rangle \to \mathbb{C}/\langle 1, \tau' \rangle$ if and only if there is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau + b}{c\tau + d}$.

Thus the space $\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$ is the "moduli space" of 1-dimensional complex tori, i.e. each point represents an isomorphism class.

Remark 11. *M*, *N* are isomorphic as complex manifolds if there are holomorphic maps $f\colon M \to N$ and $g\colon N \to M$ with $f \circ g = \mathrm{id}_N$, $g \circ f = \mathrm{id}_M$. This is sometimes called a **biholomorphic map**, but we will probably not use that term.

Linear algebra of complex structures

Definition 12. Let V be an $\mathbb{R}$-vector space. A **complex structure on** V is an endomorphism $J\colon V \to V$ with $J^2 = -I$. This turns V into a $\mathbb{C}$-vector space via $i \cdot v = J(v)$ for all $v \in V$.

Note that the minimal polynomial of J is $x^2 + 1$, so in particular J has eigenvalues $\pm i$, and $J\colon V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ is diagonalizable. Hence $V \otimes_R \mathbb{C} = V_+ \otimes V_-$, where $V_\pm$ is the J-eigenspace of $\pm i$.

Note there is a complex conjugation map

$$\begin{array}{ccc} V \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & V \otimes_{\mathbb{R}} \mathbb{C} \\ v \otimes z & \longmapsto & v \otimes \overline{z} \end{array}$$

If $v \in V_+$, then $Jv = iv$, so $\overline{Jv} = -i\overline{v}$. But J is a real operator, so $\overline{J} = J$ and we therefore have that $J\overline{v} = -i\overline{v}$. Hence, complex conjugation induces a map $V_+ \to V_-$ that is an isomorphism of $\mathbb{R}$-vector spaces. In particular,

$$\dim_{\mathbb{C}} V_+ = \dim_{\mathbb{C}} V_- = \dim_{\mathbb{C}} V.$$

Remark 13. Giving J is the same as giving the splitting

$$V \otimes_{\mathbb{R}} \mathbb{C} = V_+ \oplus V_-,$$

with $\overline{V}_+ = V_-$ and $\overline{V}_- = V_+$.

Remark 14. We also get a complex structure $J^T\colon V^* \to V^*$, giving a splitting

$$V^* \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V_+^* \oplus V_-^*$$

as before. An element of V_+^* is a **form of type** $(1,0)$**.** An element of V_-^* is a **form of type** $(0,1)$**.**

Remark 15. The point of all of this is that the tangent and cotangent spaces of a complex manifold acquire this structure.

Example 16. Let $z_1, \ldots, z_n$ be coordinates on $\mathbb{C}^n$. Write $z_j = x_j + iy_j$. The $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are real coordinates on $\mathbb{C}^n$. For $p \in \mathbb{C}^n$, the tangent space $T_p\mathbb{C}^n$ has a basis $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n$ and $T_p^*\mathbb{C}^n$ has a basis $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$.

Define $J\colon T_p\mathbb{C}^n \to T_p\mathbb{C}^n$ by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Then $J^T(dx_j) = -dy_j$ and $J^T(dy_i) = dx_j$. A basis for the $+i$ eigenspace for J^T is $dz_1, \ldots, dz_n$ where $dz_j = dx_j + idy_j$. So

$$J^T(dx_j + idy_j) = -dy_j + idx_j = i(dx_j + idy_j)$$

A basis for the $-i$ eigenspace for J^T is $d\overline{z_j} = dx_j - idy_j$.

The dual basis to $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ of $T_p^*\mathbb{C}^n \otimes \mathbb{C}$ is $\partial/\partial z_1, \ldots, \partial/\partial z_n, \partial/\partial \overline{z}_1, \ldots, \partial/\partial \overline{z}_n$, where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Remark 17 (Recall)**.** If $f\colon \mathbb{C} \to \mathbb{C}$ is a real differentiable function, then the **Cauchy-Riemann equations** say that $f = u + iv$ is holomorphic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let's apply this to Example 16.

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\left(\frac{\partial}{\partial u}x - \frac{\partial}{\partial v}y\right) + i\left(\frac{\partial}{\partial u}y + \frac{\partial}{\partial v}x\right)\right).$$

So the Cauchy-Riemann equations hold if and only if ${}^{\partial f}/_{\partial \overline{z}} = 0$. In general, $f\colon \mathbb{C}6n \to \mathbb{C}$ is holomorphic if and only if ${}^{\partial f}/_{\partial \overline{z}_i} = 0$ for all i.

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Last time, when $M = \mathbb{C}^n$, with coordinates $z_1, \ldots, z_n$ with $z_j = x_j + iy_j$, we found a basis for $T_p\mathbb{C}^n$. This basis is given by

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

The Cauchy-Riemann equations say that f is holomorphic if and only if ${}^{\partial f}/_{\partial \overline{z}_j} = 0$ for all i.

Proposition 18. An important consequence of the above. If M is a complex manifold, then there is a well-defined endomorphism of vector bundles $J\colon TM \to TM$ with $J^2 = -I$. That is, there is a complex structure on the tangent bundle.

Proof. Given a coordinate chart $\psi\colon U \to \mathbb{C}^n$, $TU = TM|_U$ inherits the endomorphism J from $T\mathbb{C}^n$ that we defined before. We need to show that this is well-defined irrespective of charts.

Note that specifying $J\colon V \to V$ with $J^2 = -I$ is equivalent to specifying a decomposition of $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ into $V_+ \oplus V_-$. Indeed, we define $J_\mathbb{C}\colon V_\mathbb{C} \to V_\mathbb{C}$ to be the linear transformation

$$J_\mathbb{C}(v_+, v_-) = (iv_+, -iv_-)$$

for $(v_+, v_-) \in V_+ \oplus V_-$. So this defines a map on $V_\mathbb{C}$, but we want an endomorphism of the real vector space V. Note that $V \subseteq V_\mathbb{C}$ is the subset invariant under conjugation, that is, consists of elements of the form $(v, \overline{v})$ with $v \in V_+$. Then $J_\mathbb{C}(v, \overline{v}) = (iv, -i\overline{v}) = (iv, \overline{iv}) \in V$. So we can define $J = J_\mathbb{C}|_V$. Therefore, specifying J is the same as specifying a splitting of $V_\mathbb{C}$.

So we just need to check that the definition is invariant under change of coordinates. We write $T_U \otimes_\mathbb{R} \mathbb{C} = T_{U,+} \oplus T_{U,-}$, and it's enough to show that the subbundles $T_{U,+}$ and $T_{U,-}$ are preserved under change of coordinates.

In particular, if $f\colon U \to \mathbb{C}^n$ is given by

$$f(z_1, \dots, z_n) = (f_1(z_1, \dots, z_n), \dots, f_n(z_1, \dots, z_n)),$$

then[1]

$$f_* \frac{\partial}{\partial z_i} = \sum_j \frac{\partial f_j}{\partial z_i} \frac{\partial}{\partial z_j} + \sum_j \frac{\partial \overline{f}_j}{\partial z_i} \frac{\partial}{\partial \overline{z}_j}$$

But ${}^{\partial \overline{f}_j}/_{\partial z_i} = \overline{{}^{\partial f_j}/_{\partial \overline{z}_i}} = 0$ since f_j is holomorphic. Therefore, the pushforward of a holomorphic vector field is just

$$f_* \frac{\partial}{\partial z_i} = \sum_j \frac{\partial f_j}{\partial z_i} \frac{\partial}{\partial z_j}$$

So $f_*(T_{U,+}) \subseteq T_{\mathbb{C}^n,+}$, and similarly $f_*(T_{U,-}) \subseteq T_{\mathbb{C}^n,-}$. This shows that the splittings are well-defined. □

Remark 19. Maybe it's easier to use the cotangent bundle conceptually, because

$$f^*(dz_i) = df_i = \sum_j \frac{\partial f_i}{\partial z_j} dz_j + \sum_j \frac{\partial f_i}{\partial \overline{z}_j} d\overline{z}_j,$$

and the second term cancels because f is holomorphic and obeys the Cauchy-Riemann equations.

Definition 20. Given a real manifold M, an **almost complex structure** on M is an endomorphism $J\colon T_M \to T_M$ with $J^2 = -I$. An almost complex structure is **integrable** if it arises from a complex (manifold) structure on M.

What we just showed is that every complex manifold has an almost complex structure, which is good, because it has a complex structure and we don't want to abuse language.

Remark 21. Integrability of an almost complex structure can be tested by the vanishing of the Nijenhuis tensor, by the Newlander-Nirenberg Theorem.

Example 22. S^6 carries an almost complex structure, but it is not known whether or not it carries a complex structure.

Definition 23. If M is an almost complex manifold, we have a splitting of the cotangent bundle as $T_M^* \otimes_{\mathbb{R}} \mathbb{C} = \Omega_M^{1,0} \oplus \Omega_M^{0,1}$ into $\pm i$-eigenspaces. Sections of $\Omega_M^{1,0}$ and $\Omega_M^{0,1}$ are called **(differential) forms of type** $(1,0)$ **and** $(0,1)$, respectively.

Recall that the vector bundle of n-forms on M is $\bigwedge^n T_M^*$. Then

$$\bigwedge^n (T_M^* \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{\substack{p,q \\ p+q=n}} \left(\bigwedge^p \Omega_M^{1,0}\right) \otimes \left(\bigwedge^q \Omega_M^{0,1}\right)$$

[1] Nobody seems to know, it's just the chain rule. Write it out in terms of the x_j and y_j.

Definition 24.
$$\Omega_M^{p,q} := \left(\bigwedge^p \Omega_M^{1,0}\right) \otimes \left(\bigwedge^q \Omega_M^{0,1}\right)$$
A section of this vector bundle is called a **form of type** (p, q).

What does such an object look like in coordinates? If $M = \mathbb{C}^n$ with the standard complex structure, (p, q) forms are given by

$$\sum_{I,J} f_{IJ} dz_I \otimes d\bar{z}_J,$$

where $I, J \subseteq \{1, \ldots, n\}$ with $\#I = p$, $\#J = q$, and

$$dz_I = \bigwedge_{i \in I} dz_i, \qquad d\bar{z}_J = \bigwedge_{j \in J} d\bar{z}_j$$

and $f_{IJ} \colon \mathbb{C}^n \to \mathbb{C}$ is C^∞. We will often write

$$\sum_{I,J} f_{IJ} dz_I \otimes d\bar{z}_J = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J.$$

So far we've been talking about smooth vector bundles, but now that we're working with complex manifolds instead of real manifolds we should think about holomorphic vector bundles. Recall that a C^∞ vector bundle over a C^∞ manifold M is a smooth manifold E along with a C^∞ map $\pi \colon E \to M$ such that there is an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i \colon U_i \to U_i \times \mathbb{R}^n$ satisfying

$$\begin{array}{ccc} \pi_i^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{R}^n \\ \big\downarrow & & \big\downarrow \text{projection} \\ U_i & \xrightarrow{\sim} & U_i \end{array}$$

with $\phi_i \circ \phi_j^{-1} \colon (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ restricts fiberwise to elements of $\mathrm{GL}_n(\mathbb{R})$.

We can change everything to say holomorphic instead.

Definition 25. A rank n holomorphic vector bundle on a complex manifold M is a complex manifold E with a holomorphic map $\pi \colon E \to M$ and biholomorphic maps $\phi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ such that

$$\begin{array}{ccc} \pi_i^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{C}^n \\ \big\downarrow & & \big\downarrow \text{projection} \\ U_i & \xrightarrow{\sim} & U_i \end{array}$$

commutes and $\phi_i \circ \phi_j^{-1}$ acts fiberwise by elements of $\mathrm{GL}_n(\mathbb{C})$.

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Definition 26. A C^∞ **section** of a real vector bundle $\pi\colon E \to M$ is a C^∞ map $\sigma\colon M \to E$ with $\pi \circ \sigma = \mathrm{id}$. Locally, on U_i, the composition

$$U_i \xrightarrow{\sigma|_{U_i}} U_i \xrightarrow{\phi_i} \pi^{-1}(U_i) \times \mathbb{R}^n \xrightarrow{\text{projection}} \mathbb{R}^n$$

is a C^∞ $\mathbb{R}^n$-valued function.

Definition 27. A **holomorphic section** of a holomorphic bundle is a holomorphic map $\sigma\colon M \to E$ with $\pi \circ \sigma = \mathrm{id}$. Similarly, the composition

$$U_i \to \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n \to \mathbb{C}^n$$

is holomorphic.

Example 28.

(1) M a complex manifold. $M \times \mathbb{C}$ is a rank 1 holomorphic bundle, with sections corresponding to holomorphic functions on M.

(2) The **holomorphic tangent bundle**. We have that $T_M \otimes_{\mathbb{R}} \mathbb{C} = T_M^+ \oplus T_M^-$ is a splitting into J-eigenspaces. Locally, T_+ has a $\mathbb{C}$-basis ${\partial}/{\partial z_1}, \ldots, {\partial}/{\partial z_n}$, and a holomorphic change of coordinates gives a *holomorphic* change of basis. So T_M^+ carries the structure of a holomorphic vector bundle, which we call the **holomorphic tangent bundle**.

Note that the same is not true of T_M^-: a holomorphic change of coordinates gives an *antiholomorphic* change of coordinates in this case.

Remark 29. The map

$$\begin{array}{ccccc} T_M & \longrightarrow & T_M \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\text{projection}} & T_M^+ \\ v & \longmapsto & v \otimes 1 & & \end{array}$$

Given locally by

$$\begin{array}{rclcl} \dfrac{\partial}{\partial x_i} & = & \dfrac{d}{dz_i} + \dfrac{\partial}{\partial \overline{z}_i} & \longmapsto & \dfrac{\partial}{\partial z_i} \\ \dfrac{\partial}{\partial y_i} & = & i\left(\dfrac{d}{dz_i} - \dfrac{\partial}{\partial \overline{z}_i}\right) & \longmapsto & \dfrac{\partial}{\partial \overline{z}_i} \end{array}$$

identifies T_M and T_M^+ as real vector bundles. With the structure of complex bundle on T_M given by J, this map is $\mathbb{C}$-linear. However, the holomorphic structure is more naturally described on T_M^+.

de Rahm cohomology

Let M be a C^∞ manifold, and let

$$A^i = \Gamma\left(M, \bigwedge^i T_M^*\right)$$

be the space of all C^∞ sections of $\bigwedge^i T_M^*$. This is just the differential i-forms on M. We get a complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots \xrightarrow{d} A^{\dim M}.$$

Write this complex as $A^\bullet$.

Definition 30. The **de Rahm cohomology** of M is

$$H^i_{\mathrm{dR}}(M,\mathbb{R}) = H^i(A^\bullet) = {\ker(A^i \xrightarrow{d} A^{i+1})}\Big/{\mathrm{im}(A^{i-1} \xrightarrow{d} A^i)}$$

Definition 31. We can also define de Rahm cohomology over $\mathbb{C}$. If we set

$$A^i_{\mathbb{C}} = \Gamma\left(M, \bigwedge^i (T_M^* \otimes_{\mathbb{R}} \mathbb{C})\right) = A^i \otimes_{\mathbb{R}} \mathbb{C},$$

then the **complex de Rahm cohomology** of M is

$$H^i_{\mathrm{dR}}(M,\mathbb{C}) = H^i(A^\bullet_{\mathbb{C}}) = H^i_{\mathrm{dR}}(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

We can also define the **Dolbeauxlt cohomology** of a complex manifold M. Recall

$$\bigwedge^n (T_M^* \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=n} \Omega_M^{p,q}$$

$$\Omega_M^{p,q} = \left(\bigwedge^p \Omega_M^{1,0}\right) \otimes \left(\bigwedge^q \Omega_M^{0,1}\right)$$

Let $A^{p,q} = \Gamma(M, \Omega_M^{p,q})$. Let $\omega \in A^{p,q}$,

$$\omega = \sum_{I,J} f_{IJ} dz_I \wedge d\overline{z}_J$$

We can take the exterior derivative.

$$d\omega = \sum_{i,I,J} \frac{\partial f_{IJ}}{\partial z_i} dz_i \wedge dz_I \wedge d\overline{z}_J + \sum_{j,I,J} \frac{\partial f_{IJ}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J \tag{2}$$

We write this as

$$d\omega = \partial\omega + \overline{\partial}\omega$$

for the two terms on the right hand side of (2). This gives maps

$$\partial\colon A^{p,q} \to A^{p+1,q}, \quad \overline{d}\colon A^{p,q} \to A^{p,q+1}$$

with $d = \partial + \overline{\partial}$. Since $d^2 = 0$,

$$(\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0$$

with

$$\partial^2\colon A^{p,q} \to A^{p+2,q}$$
$$\partial\overline{\partial}\colon A^{p,q} \to A^{p+1,q+1}$$
$$\overline{\partial}^2\colon A^{p,q} \to A^{p,q+2}$$

Thus, $\partial^2 = \overline{\partial}^2 = \overline{\partial}\partial = \partial\overline{\partial} = 0$. In particular, we have a complex $A^{p,\bullet}$

$$A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} A^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

Definition 32. We define the **Dolbeauxlt cohomology** of M to be the cohomology of this complex:

$$H^{p,q}_{\overline{\partial}}(M) = H^{p,q}(M) = H^q(A^{p,\bullet}) = \ker\left(A^{p,q} \xrightarrow{\overline{\partial}} A^{p,q+1}\right) \Big/ \mathrm{im}\left(A^{p,q-1} \xrightarrow{\overline{\partial}} A^{p,q}\right)$$

Example 33.

$$H^{p,0}_{\overline{\partial}}(M) = \ker(A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1})$$

$$\overline{\partial}\left(\sum_I f_I dz_I\right) = \sum_{j,I} \frac{\partial f_I}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I = 0$$

if and only if, for all I, j,

$$\frac{\partial f_I}{\partial \overline{z}_j} = 0$$

Thus the f_I are holomorphic. A $(p,0)$ form ω with $\overline{\partial}\omega = 0$ is called a **holomorphic p-form** and $H^{p,0}_{\overline{\partial}}(M)$ is the space of global holomorphic p-forms.

Remark 34 (Goal)**.** Most of this course will be trying to relate de Rahm cohomology to Dolbeauxlt cohomology. Our goal is to prove (modulo several weeks of hard analysis, which we will skip) the **Hodge decomposition theorem:**

$$H^n_{\mathrm{dR}}(M, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(M).$$

Some Sheaf Theory

Definition 35. Let X be a topological space. A **presheaf** $\mathcal{F}$ of abelian groups on X is the following data:

- for every open $U \subseteq X$, there is an abelian group $\mathcal{F}(U)$;
- whenever $V \subseteq U$, we have a **restriction map** $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ a group homomorphism

such that

- $\mathcal{F}(\varnothing) = 0$,
- $\rho_{U,U} = \mathrm{id}$, and
- whenever $U_3 \subseteq U_2 \subseteq U_1$, $\rho_{U_1,U_3} = \rho_{U_2,U_3} \circ \rho_{U_1,U_2}$.

Equivalently, $\mathcal{F}$ is a contravariant functor from the category of open sets on U to the category of abelian groups.

Definition 36. A **sheaf** $\mathcal{F}$ is a presheaf such that, given $U \subseteq X$ open and an open cover $\{U_i\}$ of U and

(1) if $s \in \mathcal{F}(U)$ such that $\rho_{U,U_i \cap U}(s) = 0$ for all i, then $s = 0$;

(2) if we have $s_i \in \mathcal{F}(U_i)$ for each i such that $\rho_{U_i,U_i \cap U_j}(s_i) = \rho_{U_j,U_i \cap U_j}(s_j)$, then there exists a section $s \in \mathcal{F}(U)$ with $\rho_{U,U_i}(s) = s_i$ for all i.

Lecture 5 — 21 January 2016

Example 37. Let's see some examples of sheaves.

(1) If M is a smooth manifold, define $\mathcal{F}(U) = \{f \colon U \to \mathbb{R} \text{ smooth}\}$.

(2) If $E \xrightarrow{\pi} M$ is a smooth vector bundle on M, define a sheaf $\mathcal{E}(U) := \{\sigma \colon U \to E \mid \sigma \text{ a smooth section of } \pi\}$.

(3) If M is a complex manifold, define a sheaf $\mathcal{O}_M$, called the **structure sheaf**. This is given by $\mathcal{O}_M(U) = \{f \colon U \to \mathbb{C} \text{ holomorphic}\}$.

(4) Similarly, we can define a sheaf $\mathcal{E}$ of holomorphic sections of a vector bundle $E \xrightarrow{\pi} M$.

(5) $\Omega^p_M = \bigwedge^p (T^+_M)^* = \bigwedge^p \Omega^{1,0}_M$ is a holomorphic vector bundle with sections being holomorphic differential forms, which form the associated sheaf.

(6) A non-example. Let $M = \mathbb{C}$, and let $\mathcal{B}(U)$ be the set of bounded holomorphic functions on U. The first sheaf axiom holds, but the gluing axiom fails: given an open cover of $\mathbb{C}$ by the balls $B(0,n)$ for $n \in \mathbb{N}$, the function $f(z) = z$ is locally bounded but not globally.

(7) Another non-example. Let X be a topological space. Let G be an abelian group. Define

$$\mathcal{G}^{\text{pre}}(U) = \begin{cases} G & U \neq \varnothing \\ 0 & U = \varnothing \end{cases}$$

with restriction maps identity or zero as appropriate. This is not a sheaf, because the sheaf gluing axiom fails in the case that $U = U_1 \sqcup U_2$. If $\mathcal{G}^{\text{pre}}$ was a sheaf, then for any $g_1, g_2 \in G$, there should be some $g \in \mathcal{G}^{\text{pre}}$ with $g|_{U_1} = g_1$ and $g|_{U_2} = g_2$. But this cannot happen, because $U_1 \cap U_2 = \varnothing$ and $\mathcal{G}^{\text{pre}}(\varnothing) = 0$.

We want a definition that kind of looks like the last example, but is actually a sheaf. This is the following definition.

Definition 38. The **constant sheaf** $\mathcal{G}$ can be defined by putting the discrete topology on G and defining $\mathcal{G}(U) = \{f \colon U \to G \text{ continuous}\}$. These maps are **locally constant**, i.e. constant on connected components.

This might seem like a relatively stupid sheaf, but it'll be quite important for us.

Definition 39. Let $\mathcal{F}$ be a sheaf on a space X. The **stalk of $\mathcal{F}$ at p** is

$$\mathcal{F}_p := \big\{(U,s) \mid U \ni p \text{ open}, s \in \mathcal{F}(U)\big\} / \sim$$

where $(U,s) \sim (V,t)$ if there is some $W \subseteq U \cap V$ such that $s|_W = t|_W$. Elements (U,s) of the stalk $\mathcal{F}_p$ are called **germs** of the function s at p.

Example 40. Let M be a complex manifold. Then $\mathcal{O}_{M,p} \cong \mathbb{C}\{z_1, \ldots, z_n\}$ is the ring of convergent power series in a neighborhood of 0 in $\mathbb{C}^n$.

Definition 41. A **morphism** $f\colon \mathcal{F} \to \mathcal{G}$ **of sheaves on** X is a collection of group homomorphisms $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ with some compatibility conditions with respect to restriction. Namely, given $V \subseteq U$, the following commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{U,V}} & & \downarrow{\scriptstyle \rho_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

Remark 42. A sheaf morphism $f\colon \mathcal{F} \to \mathcal{G}$ induces a map of stalks

$$\begin{array}{ccc} \mathcal{F}_p & \xrightarrow{f_p} & \mathcal{G}_p \\ (U,s) & \longmapsto & (U, f(U)(s)) \end{array}$$

Definition 43. A sequence of sheaves and morphisms

$$\cdots \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \cdots$$

is **exact** if the corresponding sequence

$$\cdots \longrightarrow (\mathcal{F}_{i-1})_x \longrightarrow (\mathcal{F}_i)_x \longrightarrow (\mathcal{F}_{i+1})_x \longrightarrow \cdots$$

is exact for all $x \in X$.

Remark 44 (Warning!). Definition 43 is *not* the same as the sequence

$$\cdots \longrightarrow \mathcal{F}_{i-1}(U) \longrightarrow \mathcal{F}_i(U) \longrightarrow \mathcal{F}_{i+1}(U) \longrightarrow \cdots$$

being exact.

Example 45. Hopefully this will be your favorite example by the time we're done with the course. Let M be a complex manifold, $\mathbb{Z}$ the constant sheaf given by $G = \mathbb{Z}$, $\mathcal{O}_M$ the structure sheaf, $\mathcal{O}_M^*$ the sheaf of invertible holomorphic functions on M. Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O}_M \xrightarrow{e} \mathcal{O}_M^* \longrightarrow 0$$

Here, i is just the inclusion map, and e is the map $f \mapsto \exp(2\pi i f)$. This is called the **exponential exact sequence**.

Let's check that this is exact. The injectivity of the inclusion map i is clear.

To see exactness in the middle, suppose $(U, f) \in \mathcal{O}_{M,p}$. Then the germs $(U, \exp(2\pi i f))$ and $(U, 1)$ are equivalent if and only if f is constant in a neighborhood of p if and only if f is in the image of the inclusion map i.

Exactness on the right is slightly more interesting. Let (U, f) be a germ of an invertible function. We can shrink U and assume that U is simply connected. Then a branch of $\frac{1}{2\pi i} \log f$ can be chosen. This gives a germ $\left(U, \frac{1}{2\pi i} \log f\right)$ mapping to (U, f). So e is surjective.

Remark 46. Surjectivity has to be tested on stalks; the map $\mathcal{O}_M(U) \to \mathcal{O}_M(U)^*$ need not be surjective. For example, take $M = \mathbb{C} \backslash \{0\}$, and $f(z) = z$. So $\frac{1}{2\pi i} \log f$ cannot be defined globally on M.

Proposition 47. The sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G}$$

is exact if and only if the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)$$

is exact for all $U \subseteq X$ open.

Proof. ($\Longrightarrow$). Assume $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is injective, which means that $\mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Let $s \in \mathcal{F}(U)$ with $f(U)(s) = 0$. Then $(U, s) \in \mathcal{F}_x$ maps to $(U, 0)$ in $\mathcal{G}_x$. So by injectivity, $(U, s) = 0$ in $\mathcal{F}_x$. So there is a neighborhood $U_x \subseteq U$ with $x \in U_x$ and $s|_{U_x} = 0$. This holds for each $x \in U$, so we have an open cover $\{U_x \mid x \in U\}$ of U and therefore $s = 0$ by the first sheaf axiom.

($\Longleftarrow$). If $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subseteq X$, let $(U, s) \in \mathcal{F}_x$ with $(U, f(U)(s)) = 0$ in $\mathcal{G}_x$. So there is some $V \subseteq U$ with $x \in V$, $f(U)(s)|_V = 0$. But $f(U)(s)|_V = f(V)(s|_V)$, so $s|_V = 0$. Hence, $(U, s) \sim (V, s|_V) = 0$ in $\mathcal{F}_x$. ☐

Proposition 48. If

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3$$

is exact, so is

$$0 \to \mathcal{F}_1(U) \xrightarrow{f(U)} \mathcal{F}_2(U) \xrightarrow{g(U)} \mathcal{F}_3(U)$$

for all $U \subseteq X$.

Proof. Injectivity on the left is Proposition 47.

$\operatorname{im}(f(U)) \subseteq \ker(g(U))$ is equivalent to $g(U) \circ f(U) = 0$. But $(U, (g(U) \circ f(U))(s)) \in (\mathcal{F}_3)_x$ is zero, if $s \in \mathcal{F}_1(U)$ by exactness at the stalk level, for all $x \in U$. So we can use the same trick as before to see that $(g(U) \circ f(U))(s) = 0$.

Conversely, to show that $\ker(g(U)) \subseteq \operatorname{im}(f(U))$, let $s \in \mathcal{F}_2(U)$ such that $g(U)(s) = 0$. For each $x \in U$, there is $U_x \ni x$ open and a germ $(U_x, t_x) \in (\mathcal{F}_1)_x$ such that $(U_x, f(U_x)(t_1)) = (U, s) \in (\mathcal{F}_2)_x$. By shrinking U_x if necessary, we may assume that $s|_{U_x} = f(U_x)(t_x)$. So now we have an open cover $\{U_x \mid x \in U\}$ of U and sections over U_x. Let's see what they do on the overlaps:

$$f(U_x \cap U_y)(t_x - t_y)|_{U_x \cap U_y} = (s - s)|_{U_x \cap U_y} = 0$$

But we have already shown that $f(U_x \cap U_y)$ is injective, so $(t_x - t_y)|_{U_x \cap U_y} = 0$. hence, the sections t_x glue to give some $t \in \mathcal{F}_1(U)$ with $f(U)(t) = s$. ☐

Lecture 6 — 23 January 2016

Last time we talked about injectivity and surjectivity of sheaf maps. Injectivity is easy, because injectivity on stalks holds if and only if injectivity on every open set holds, but the same is not true for surjectivity. This makes sheaf kernels natural but sheaf cokernels are kind of gross.

Definition 49. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then we define the **sheaf kernel**, which is a sheaf denoted by $\ker f$, to be

$$(\ker f)(U) = \ker(\{(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)).$$

Exercise 50. Check that this is a valid sheaf.

If we want to do the same thing for the cokernel, we run into trouble.

Definition 51. The **sheaf cokernel** of $f\colon \mathcal{F} \to \mathcal{G}$ is defined by

$$(\operatorname{coker} f)(U) = \left\{ \{(U_i, s_i)\} \;\middle|\; \begin{array}{c} \{U_i\} \text{ is an open cover of } U, s_i \in \mathcal{G}(U_i), \text{ s.t.} \\ \text{for all } i, j, s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \operatorname{im} f(U_i \cap U_j) \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation $\{(U_i, s_i)\} \sim \{(U_i', s_i')\}$ if for all $x \in U$, $x \in U_i \cap U_j'$, there is $V \subseteq U_i \cap U_j'$ with $x \in V$ such that $s_i|_V = s_j'|_V \in \operatorname{im} f(V)$.

Exercise 52. Show that $\operatorname{coker} f$ is a sheaf, and show that

$$0 \longrightarrow \ker f \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \operatorname{coker} f \longrightarrow 0$$

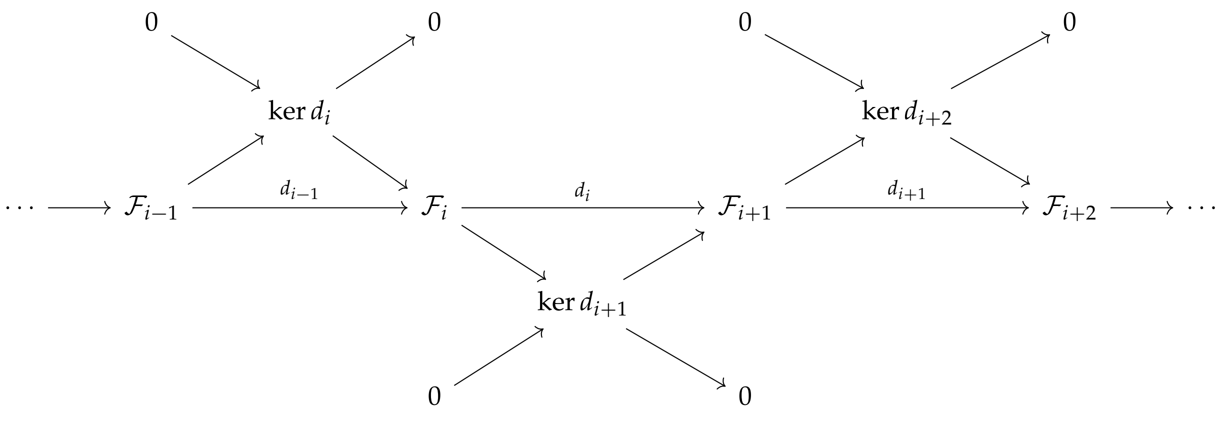
is exact.

Remark 53. This definition makes the set of sheaves of abelian groups over a given topological space into an abelian category.

Remark 54. If we have a long exact sequence

$$\cdots \longrightarrow \mathcal{F}_{i-1} \xrightarrow{d_{i-1}} \mathcal{F}_i \xrightarrow{d_i} \mathcal{F}_{i+1} \longrightarrow \cdots$$

splits up into a collection of short exact sequences



with all diagonal sequences exact.

Definition 55 (Notation)**.** We often write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$. Note that $\Gamma(U, \cdot)$ is a functor.

Cohomology of Sheaves

Given a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

we have a short exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}_1) \longrightarrow \Gamma(X, \mathcal{F}_2) \longrightarrow \Gamma(X, \mathcal{F}_3) \longrightarrow 0.$$

Definition 56. A sheaf $\mathcal{F}$ is **flabby (flasque)** if whenever $V \subseteq U$, $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Theorem 57 (Sheaf Cohomology)**.** There are contravariant functors $H^i(X, -)$ from the category of sheafs of abelian groups on X to the category of abelian groups such that

(1) whenever there is a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, there are maps $\delta \colon H^i(X, \mathcal{F}_3) \to H^{i+1}(X, \mathcal{F}_1)$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X,\mathcal{F}_1) & \longrightarrow & H^0(X,\mathcal{F}_2) & \longrightarrow & H^0(X,\mathcal{F}_3) \\ & \xrightarrow{\delta} & H^1(X,\mathcal{F}_1) & \longrightarrow & H^1(X,\mathcal{F}_2) & \longrightarrow & H^1(X,\mathcal{F}_3) \\ & \xrightarrow{\delta} & H^2(X,\mathcal{F}_1) & \longrightarrow & \cdots & & \end{array}$$

is exact, with unlabelled arrows the functorial maps.

(2) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

(3) If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_3 & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows, then

$$\begin{array}{ccc} H^i(X,\mathcal{F}_3) & \xrightarrow{\delta} & H^{i+1}(X,\mathcal{F}_1) \\ \downarrow & & \downarrow \\ H^i(X,\mathcal{G}_3) & \xrightarrow{\delta} & H^{i+1}(X,\mathcal{G}_1) \end{array}$$

commutes as well, where the vertical maps are the functorial maps.

(4) If $\mathcal{F}$ is flabby, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Furthermore, H^i is uniquely determined by these properties.

Definition 58. A sheaf $\mathcal{F}$ is **acyclic** if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

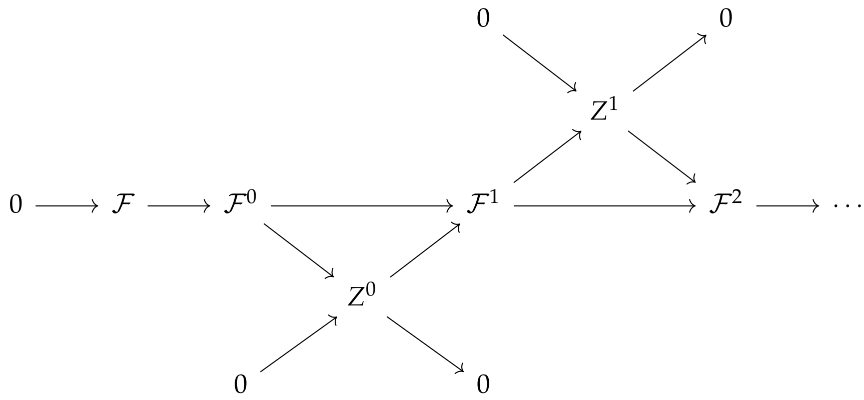
Proposition 59. Let

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

be exact with all $\mathcal{F}^i$ acyclic. This is called an **acyclic resolution** of $\mathcal{F}$, written as $0 \to \mathcal{F} \to \mathcal{F}^\bullet$. Then

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{F}^\bullet)) = \frac{\ker\left(\Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1})\right)}{\operatorname{im}\left(\Gamma(X, \mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{F}^i)\right)}$$

Proof. We sill use properties (1)-(3) of Theorem 57 as well as acyclicity. Let $Z^i = \ker\left(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2}\right)$. We have



which gives exact sequences

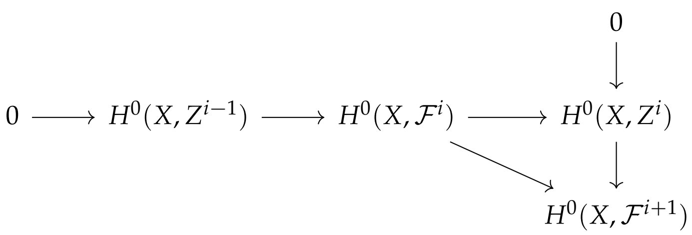
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathbb{Z}^0 \longrightarrow 0$$

$$0 \longrightarrow Z^i \longrightarrow \mathcal{F}^{i+1} \longrightarrow Z^{i+1} \longrightarrow 0$$

for all $i \geqslant 0$. For $j > 0$,

$$0 = H^j(X, \mathcal{F}^i) \to H^j(X, Z^i) \xrightarrow{\delta} H^{j+1}(X, Z^{i-1}) \to H^{j+1}(X, \mathcal{F}^i) = 0$$

so δ is an isomorphism. Similarly, we get that $H^j(X, Z^0) \cong H^{j+1}(X, \mathcal{F})$. We also have that the following commutes,



with the bottom exact. So

$$H^0(X, Z^{i-1}) = \ker(H^0(X, \mathcal{F}^i) \to H^0(X, \mathcal{F}^{i+1}))$$

and similarly,

$$H^0(X,\mathcal{F}) = \ker(H^0(X,\mathcal{F}^0) \to H^0(X,\mathcal{F}^1))$$

and we also have that

$$H^0(X,\mathcal{F}^i) \longrightarrow H^0(X,Z^i) \longrightarrow H^1(X,Z^{i-1}) \longrightarrow H^1(X,\mathcal{F}^i) = 0$$

is exact. Thus,

$$H^1(X,Z^{i-1}) = \operatorname{coker}(H^0(X,\mathcal{F}^i) \to H^0(X,Z^1)) = \frac{\ker(H^0(X,\mathcal{F}^{i+1} \to H^0(X,\mathcal{F}^{i+2}))}{\operatorname{im}\left(H^0(X,\mathcal{F}^i) \to H^0(X,\mathcal{F}^{i+1})\right)} = H^{i+1}(\Gamma(X,\mathcal{F}^\bullet))$$

But

$$H^1(X,Z^{i-1}) \cong H^2(X,Z^{i-2}) \cong \ldots \cong H^i(X,Z^0) \cong H^{i+1}(X,\mathcal{F})$$

Combining the previous two lines gives the desired result. □

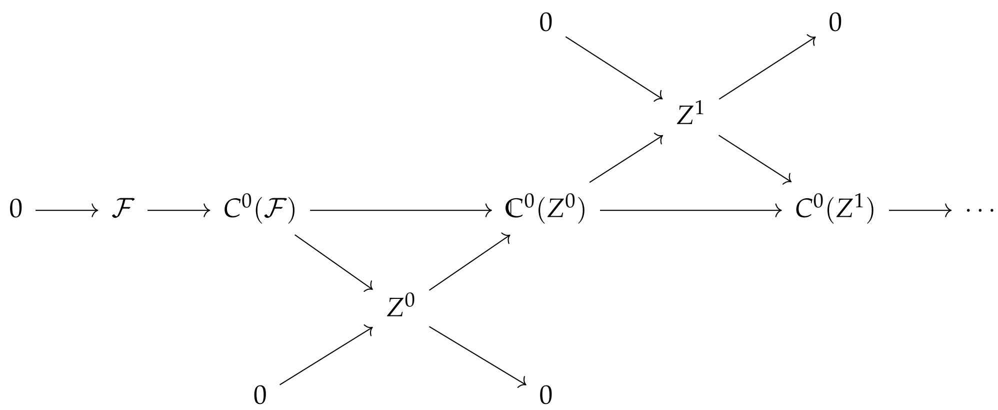
Example 60. Given a sheaf $\mathcal{F}$, define $C^0(\mathcal{F})$ to be the sheaf

$$C^0(\mathcal{F})(U) = \{f\colon U \to \coprod_{p\in U} \mathcal{F}_p \mid f(p) \in \mathcal{F}_p\}.$$

This sheaf is flabby. There is an inclusion

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & C^0(F) \\ & & s \in \mathcal{F}(U) & \longmapsto & \left(U \ni p \mapsto (U, s(p)) \in \mathcal{F}_p\right) \end{array}$$

Then



gives a flabby resolution of $\mathcal{F}$. As far as I know, nobody has ever used this to do a computation.

Lecture 7

26 January 2016

Last time we talked about the cohomology of sheaves. You should take it as a black box and just use the theorems for computations.

Čech cohomology

Definition 61. Let $\mathcal{F}$ be a sheaf on X, and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering on an ordered set I. Define

$$C^0(\mathcal{U},\mathcal{F}) := \prod_{i\in I} \mathcal{F}(U_i)$$

$$C^1(\mathcal{U},\mathcal{F}) := \prod_{\substack{i_0<i_1\\ i_i,i_1\in I}} \mathcal{F}(U_{i_0}\cap U_{i_1})$$

$$C^p(\mathcal{U},\mathcal{F}) := \prod_{\substack{i_0<\ldots<i_p\\ i_0,\ldots,i_p\in I}} \mathcal{F}(U_{i_0}\cap \ldots \cap U_{i_p})$$

the group of **Čech p-cochains**.

Define boundary maps $\delta\colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ by

$$(\delta\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}}$$

for $\alpha \in C^p(\mathcal{U},\mathcal{F})$.

One can check that $\delta^2 = 0$. Hence, get a complex $C^\bullet(\mathcal{U},\mathcal{F})$ defined by

$$C^0(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U},\mathcal{F}) \to \cdots$$

The cohomology of this complex is **Čech cohomology**:

$$\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) = H^p(\mathbb{C}^\bullet(\mathcal{U},\mathcal{F})) = \frac{\ker\left(C^p \xrightarrow{\delta} C^{p+1}\right)}{\operatorname{im}\left(C^{p-1} \xrightarrow{\delta} C^p\right)}.$$

$$\begin{aligned}
\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) &= \ker\left(C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})\right) \\
&\qquad \{(U_i, s_i) \mid i \in I\} \mapsto \left\{(U_i \cap U_j, s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}) \mid i < j\right\} \\
&= H^0(X,\mathcal{F}) && \text{by sheaf axioms} \\
&= \Gamma(X,\mathcal{F})
\end{aligned}$$

Definition 62. For shorthand, write $U_{i_0,\ldots,i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$.

Remark 63. We have a problem here, namely that $\check{\mathrm{H}}(\mathcal{U},\mathcal{F})$ depends on the open cover, for example, if $\mathcal{U} = \{X\}$ then there would be no cohomology higher than $\check{\mathrm{H}}^0$. So this theory would be useless.

Definition 64. Let $\mathcal{U},\mathcal{U}'$ be open covers with a increasing map $\phi\colon I' \to I$ between index sets, with $U'_i \subseteq U_{\phi(i)}$. In this case we write $\mathcal{U}' < \mathcal{U}$. This defines a map

$$\rho_\phi\colon C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U}',\mathcal{F})$$

$$(\rho_\phi \alpha)_{i_0,\ldots,i_p} = \alpha_{\phi(i_0),\ldots,\phi(i_p)}|_{U'_{i_0,\ldots,i_p}}$$

for $i_0, \ldots, i_p \in I'$. This gives $\delta \circ \rho_\phi = \rho_\phi \circ \delta$, and hence we get a map on cohomology

$$\rho_\phi \colon \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{U}', \mathcal{F}).$$

Definition 65.

$$\check{\mathrm{H}}^p(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}),$$

where the limit is taken over all open covers under the relation $<$.

Definition 66 (Recall)**.** Given a partially ordered set I, and a system of abelian groups G_i for all $i \in I$ with maps $\rho_{ij} \colon G_i \to G_j$ for $i < j$ with $\rho_{ij} \circ \rho_{jk} = \rho_{ik}$, then the direct limit of the groups G_i over i is

$$\varinjlim_{i \in I} G_i := \bigoplus_{i \in I} G_i \Big/ N$$

where N is the subgroup of the direct sum generated by elements $(a_i \mid i \in I)$ of the form

$$a_i = \begin{cases} -\rho_{kj}(g_j) & i = k \\ g_j & i = j \\ 0 & \text{else} \end{cases}.$$

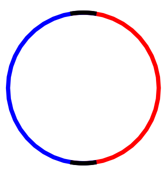
The moral of this story is that elements of $\check{\mathrm{H}}^p(X, \mathcal{F})$ are represented by

$$\{(U_{i_0,\ldots,i_p}, s_{i_0,\ldots,i_p})\} \in \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}).$$

Two different elements are compared over refinements. This makes it easy to get our hands on representatives of cohomology classes, and it was not obvious how to do so before. The important theorem is that this is the same as regular cohomology of X.

Theorem 67. $\check{\mathrm{H}}(X, \mathcal{F}) = H^p(X, \mathcal{F})$

Example 68. $X = S^1$, $\mathcal{F}$ the constant sheaf $\mathbb{Z}$. $\mathcal{U} = \{U_1, U_2\}$



U_1 is the union of red and black, U_2 is union of blue and black.

$$C^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z},$$

each component of the direct sum corresponding to an open set.

$$C^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}$$

The map $\delta\colon C^0 \to C^1$ is given by

$$\delta(a,b) = (b-a, b-a).$$

Therefore,

$$H^0(X,\mathbb{Z}) = \check{\mathrm{H}}^0(\mathcal{U},\mathbb{Z}) = \mathbb{Z}$$
$$H^1(X,\mathbb{Z}) = \check{\mathrm{H}}^1(\mathcal{U},\mathbb{Z}) = \mathbb{Z}$$

This last line follows from the next fact.

Fact 69. If G is an abelian group and $\mathcal{U}$ is an open cover of X with all $U_{i_0,\ldots,i_p}$ contractible, then $\check{\mathrm{H}}^p(\mathcal{U},G) = \check{\mathrm{H}}^p(X,G) = H^p(X,G)$.

Remark 70. In many nice cases, including when X is a manifold, then $H^p(X,G) \cong H^p_{\mathrm{sing}}(X,G)$.

Remark 71. The construction of the connecting homomorphism

$$\check{\mathrm{H}}^p(X,\mathcal{F}_3) \longrightarrow \check{\mathrm{H}}^{p+1}(X,\mathcal{F}_1)$$

for a short exact sequence $0 \to \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3 \to 0$. Let $\{(U_{i_0,\ldots,i_p}, s_{i_0,\ldots,i_p})\}$ represent an element of $\check{\mathrm{H}}^p(X,\mathcal{F}_3)$. Here $s_{i_0,\ldots,i_p} \ni \mathcal{F}_3(U_{i_0,\ldots,i_p})$. Possibly after refining the cover $\mathcal{U}$, we can lift $s_{i_0,\ldots,i_p}$ to some $t_{i_0,\ldots,i_p} \in \mathcal{F}_2(U_{i_0,\ldots,i_p})$.

Now consider

$$\delta(\{(U_{i_0,\ldots,i_p}, t_{i_0,\ldots,i_p})\}) \in C^{p+1}(\mathcal{U},\mathcal{F}_2)$$

and

$$g(\delta(\{(U_{i_0,\ldots,i_p}, t_{i_0,\ldots,i_p})\})) = \delta(\{(U_{i_0,\ldots,i_p}, s_{i_0,\ldots,i_p})\}) = 0$$

Thus there exists $t'_{i_0,\ldots,i_{p+1}} \in \mathcal{F}_1(U_{i_0,\ldots,i_{p+1}})$ with

$$f(t'_{i_0,\ldots,i_{p+1}}) = \delta\left(\{(U_{i_0,\ldots,i_p}, t_{i_0,\ldots,i_p})\}\right)|_{i_0,\ldots,i_{p+1}}$$

Then $\{(U_{i_0,\ldots,i_{p+1}}, t'_{i_0,\ldots,i_{p+1}})\} \in C^{p+1}(\mathcal{U},\mathcal{F}_1)$ represents an element of $\check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F})$.

Lecture 8 — 30 January 2016

Last time we were talking about Čech cohomology, but today we'll come down to earth with a very explicit example.

Example 72. Let

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 1$$

be the exponential exact sequence, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M and $\mathcal{O}_M^*$ is the nowhere vanishing holomorphic functions. We have connecting homomorphisms $\mathrm{H}^0(M,\mathcal{O}_M^*) \to \mathrm{H}^1(M,\mathbb{Z})$ and $\mathrm{H}^1(M,\mathcal{O}_M^*) \to \mathrm{H}^2(M,\mathbb{Z})$. These are sheaf cohomology, but it's the same as singular cohomology that you may have seen in algebraic topology.

An element $s \in \mathrm{H}^0(M, \mathcal{O}_M^*)$ has a Čech representative $\{(M, s)\}$ where the open cover is $\mathcal{U} = \{M\}$. By passing to an open cover $\mathcal{U} = \{U_i\}$ of M with U_i simply connected, then $\{(M, s)\}$ is identified with $\{(U_i, s|_{U_i})\} \in \mathrm{H}^0(\mathcal{U}, \mathcal{O}_M^*)$.

On U_i, there is a holomorphic function $g_i = \frac{1}{2\pi i}\log(s|_{U_i})$ (after choosing a branch appropriately). This gives a Čech cochain $\{(U_i, g_i)\}$ for $\mathcal{O}_M$, with coboundary

$$\alpha = \{(U_i \cap U_j, g_j|_{U_i \cap U_j} - g_i|_{U_i \cap U_j}) \mid i < j\} \in \mathrm{H}^1(\mathcal{U}, \mathbb{Z}).$$

We're trying to do something very simple here: understand the obstruction to lifting these locally defined logarithms.

Suppose this cocycle vanishes in $\check{\mathrm{H}}^1(\mathcal{U}, \mathbb{Z})$. Then there exists $\beta = \{(U_i, a_i)\} \in \mathrm{C}^0(\mathcal{U}, \mathbb{Z})$ such that $\delta(\beta) = \alpha$, i.e. on $U_i \cap U_j$, we have

$$g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j} = a_j - a_i.$$

Then set $g_i' = g_i - a_i$. We have then that

$$g_i' - g_j' = (g_i|_{U_i \cap U_j} - a_i) - (g_j|_{U_i \cap U_j} - a_j) = 0.$$

Thus the g_i' glue to give $g' \in \mathrm{H}^0(M, \mathcal{O}_M)$ with $\frac{2\pi i}{\log} g' = s$.

We just showed exactness at $\mathrm{H}^0(M, \mathcal{O}_M^*)$ in the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(M, \mathbb{Z}) \longrightarrow \mathrm{H}^0(M, \mathcal{O}_M) \longrightarrow \mathrm{H}^0(M, \mathcal{O}_M^*) \xrightarrow{\delta} \mathrm{H}^1(M, \mathbb{Z}) \longrightarrow \cdots$$

What does a Čech representative for $\mathrm{H}^1(M, \mathcal{O}_M^*)$ really mean? This is a collection

$$\alpha = \big\{(U_i \cap U_j, g_{ij}) \mid i < j\big\}$$

with $g_{ij} \in \mathcal{O}_M^*(U_i \cap U_j)$ and $g_{ij} \colon U_i \cap U_j \to \mathbb{C}^\times$ holomorphic. Note that $\delta(\alpha) = 0$ if and only if $g_{ij} g_{ik}^{-1} g_{jk} = 1$ for all $i < j < k$ on $U_i \cap U_j \cap U_k$ or $g_{ij} \cdot g_{jk} = g_{ik}$.

If $\delta(\alpha) = 0$, this defines a holomorphic rank 1 vector bundle, that is, a holomorphic line bundle. Such a line bundle $\mathcal{L}$ is given on U_i by $\mathcal{L}_i = U_i \times \mathbb{C}$ and by transition maps

$$\begin{array}{ccc} (U_i \cap U_j) \times \mathbb{C} & \xrightarrow{g_{ij}} & (U_i \cap U_j) \times \mathbb{C} \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{L}_i|_{U_i \cap U_j} & & \mathcal{L}_j|_{U_i \cap U_j} \end{array}$$

where the arrow labelled g_{ij} represents the map $g_{ij}(p, z) = (p, g_{ij}(p) \cdot z)$. Consistency of identifications is given by $g_{ij} \cdot g_{jk} = g_{ik}$, so these locally trivial bundles glue to give a line bundle on the whole manifold.

Exercise 73. (1) Given two Čech representatives for the same cohomology class in $\mathrm{H}^1(M, \mathcal{O}_M^*)$, then they give isomorphic line bundles under this construction.

(2) Show the converse: if two Čech cocycles give isomorphic line bundles, then these Čech cocycles define the same cohomology class.

Definition 74. The **Picard group** of M is the group $\operatorname{Pic}(M)$ of holomorphic line bundles on M modulo isomorphism. By the previous exercise Exercise 73, note that $\operatorname{Pic}(M) = \mathcal{H}^1(M, \mathcal{O}_M^*)$, which has a group structure.

Remark 75. If E, F are vector bundles over M, we define the tensor product $E \otimes F$ to e the vector bundle (taking $\mathcal{U} = \{U_i\}$ a cover of M which trivializes both E and F), which on U_i is $U_i \times (\mathbb{C}^n \otimes \mathbb{C}^m)$ and transition maps

$$(U_i \cap U_j) \times (\mathbb{C}^n \otimes \mathbb{C}^m) \longrightarrow (U_i \cap U_j) \times (\mathbb{C}^m \otimes \mathbb{C}^n)$$

are $e_{ij} \otimes f_{ij}$ where e_{ij} and f_{ij} are the transition maps for E and F, respectively.

So if $\alpha_1, \alpha_2 \in \mathrm{H}^1(M, \mathcal{O}_M^*)$ correspond to line bundles $\mathcal{L}_1, \mathcal{L}_2$, then $\alpha_1 \cdot \alpha_2$ corresponds to $\mathcal{L}_1 \otimes \mathcal{L}_2$.

Remark 76. If E is a vector bundle with transition maps e_{ij} on $U_i \cap U_j$, then $E^\vee$ is the vector bundle with transition maps $(e_{ij}^{-1})^T$. Why is there a transpose in there? We want the transitions to be compatible: on $U_i \cap U_j \cap U_k$, $e_{ij}e_{jk} = e_{ik}$, so

$$(e_{ij}^{-1})^T (e_{ij}^{-1})^T = (e_{ij}^{-1})^T.$$

The conclusion is that on $\operatorname{Pic} M$, there is a natural group operation given by $\otimes$ and inverse given by duals.

Where does the Picard group fit in with the other cohomology? We have the long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(M,\mathbb{Z}) \to \mathrm{H}^0(M,\mathcal{O}_M) &\longrightarrow \mathrm{H}^0(M,\mathcal{O}_M^*) \xrightarrow{\delta} \\ \mathrm{H}^1(M,\mathbb{Z}) \to \mathrm{H}^1(M,\mathcal{O}_M) &\to \mathrm{H}^1(M,\mathcal{O}_M^*) = \operatorname{Pic}(M) \xrightarrow{c_1} \\ \mathrm{H}^2(M,\mathbb{Z}) &\longrightarrow \cdots \end{aligned}$$

We call this map c_1 the **first Chern class map**. If M is connected, then $\mathrm{H}^0(M, \mathbb{Z}) = \mathbb{Z}$. If M is compact, then $\mathrm{H}^0(M, \mathcal{O}_M) = \mathbb{C}$ is the constant functions (*Proof:* Let $f\colon M \to \mathbb{C}$ holomorphic. Since M is compact, $|f|$ attains its maximum on M, say at $p \in M$. Passing to an open neighborhood of M, we can assume M is an open ball in $\mathbb{C}^n$, and $|f|$ realizes its maximum on the boundary by the maximum modulus principle, hence f is constant.)

Hence, we see that the first row of the diagram above becomes a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp(\frac{1}{2\pi i} z)} \mathbb{C}^\times \longrightarrow 0$$

and we may as well start our exact sequence on the second line.

So if M is compact, connected, we have an exact sqeuence

$$0 \to \mathrm{H}^1(M,\mathbb{Z}) \to \mathrm{H}^1(M,\mathcal{O}_M) \to \operatorname{Pic} M \to \mathrm{H}^2(M,\mathbb{Z}).$$

Lecture 9 — 2 February 2016

Comparison of sheaf and de Rahm cohomology

Consider the complex on a C^∞ manifold M

$$0 \to \mathbb{R} \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}_M \to 0,$$

where Ω^0_M is the sheaf of C^∞ functions on M. This is an exact sequence of sheaves: exactness at Ω^0_M is just the statement that $df = 0$ if and only if f is locally constant and exactness at Ω^p_M is the Poincaré Lemma.

Lemma 77 (Poincaré Lemma)**.** *If U is a contractible open neihborhood of $x \in M$, and α is a p-form on U, then $d\alpha = 0$ if and only if there is a $p-1$ form ω on U with $d\omega = \alpha$.*

That is, $\Omega^\bullet_M$ is a resolution of $\mathbb{R}$. If we know that $\mathrm{H}^i(M, \Omega^j_M) = 0$ for $i > 0$, for all j, then $\Omega^\bullet_M$ is an acyclic resolution of $\mathbb{R}$, and

$$\mathrm{H}^i(M, \mathbb{R}) = \mathrm{H}^i(\Gamma(M, \Omega^\bullet_M)) = \mathrm{H}^i_{\mathrm{DR}}(M, \mathbb{R}).$$

Definition 78. Let $\mathcal{F}$ be a sheaf on a space X, and let $\mathcal{U} = \{U_i\}$ be an open cover of X which is locally finite. A **partition of unity** of $\mathcal{F}$ subordinate to $\mathcal{U}$ is a collection of sheaf homomorphisms $\eta_i \colon \mathcal{F} \to \mathcal{F}$ such that

(1) η_i is zero on an open neighborhood of $X \backslash U_i$;

(2) $\sum_i \eta_i = \mathrm{id}$.

Definition 79. A sheaf is said to be **fine** if it admits a partition of unity subordinate to any locally finite cover.

Example 80. Let F be a C^∞ vector bundle on a C^∞ manifold M, and let $\mathcal{F}$ be the sheaf of C^∞ sections of F, and $\mathcal{U}$ a locally finite cover, $\{\phi_i\}$ a partition of unity in the usual sense for $\mathcal{U}$ (meaning that $\phi_i \colon M \to \mathbb{R}$ are smooths, ϕ_i is zero outside an open neighborhood of $X \backslash U_i$, and $\sum_i \phi_i = 1$).

Define $\eta_i(s) = \phi_i \cdot s$.

Lemma 81. *If $\mathcal{F}$ is a fine sheaf on a paracompact space X, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. By paracompactness, every open cover of X has a locally finite refinement $\mathcal{U}$. Let $\alpha \in C^P(\mathcal{U}, \mathcal{F})$ with $\delta\alpha = 0$. Define, for $i_0 < \ldots < i_{p-1}$,

$$\tau_{i_0,\ldots,i_{p-1}} = \sum_{j \in J} \eta_j(\alpha_{j,i_0,\ldots,i_{p-1}}).$$

Each $\eta_j(\alpha_{j,i_0,\ldots,i_{p-1}})$ lives on $U_j \cap U_{i_0} \cap \ldots \cap U_{i_{p-1}}$. Moreover, $\eta_j(\alpha_{j,i_0,\ldots,i_{p-1}})$ is zero on an open neighborhood of $U_{i_0} \cap \ldots \cap U_{i_{p-1}} \backslash U_j$ inside of $U_{i_0} \cap \ldots \cap U_{i_{p-1}}$ and hence we can extend the section by 0 to $U_{i_0} \cap \ldots \cap U_{i_{p-1}}$ by the sheaf gluing axiom. Thus,

$$(\tau_{i_0,\ldots,i_{p-1}})_{i_0,\ldots,i_{p-1}} \in C^{p-1}(\mathcal{U}, \mathcal{F}).$$

Now we have to compute $\delta\tau$:

$$\begin{aligned}(\delta\tau)_{i_0,\ldots,i_p} &= \sum_{k=0}^{p}(-1)^k\sum_{j\in J}\eta_j(\alpha_{j,i_0,\ldots,\widehat{i_k},\ldots i_p})|_{U_{i_0}\cap\ldots\cap U_{i_p}} \\ &= \sum_{j\in J}\eta_j\left(\sum_{k=0}^{p}(-1)^k\alpha_{j,i_0,\ldots,\widehat{i_k},\ldots i_p}|_{U_{i_0}\cap\ldots\cap U_{i_p}}\right) \end{aligned} \tag{3}$$

But since $\delta\alpha = 0$, then

$$\alpha_{i_1,\ldots,i_p} = \sum_{k=0}^{p}\alpha_{j,i_0,\ldots,\widehat{i_k},\ldots,i_p} = 0$$

on $U_j \cap U_{i_0} \cap \ldots \cap U_{i_p}$. This implies that (3) is equal to

$$(3) = \sum_{j\in J}\eta_j(\alpha_{i_0,\ldots,i_p})$$

□

An immediate consequence of this is the following.

Theorem 82. $\mathrm{H}^i_{\mathrm{dR}}(M,\mathbb{R}) = \mathrm{H}^i(M,\mathbb{R})$.

This, so far, was just a warm up. We really want to do this for Dolbeauxlt cohomology.

Theorem 83 ($\overline{\partial}$-Poincaré Lemma or Dolbeault-Grothendieck Lemma)**.** Let $\Delta(r) = \{(z_1,\ldots,z_n)\in\mathbb{C}^n \mid |z_i| < r\}$. Then $\mathrm{H}^{p,q}_{\overline{\partial}}(\Delta(r)) = 0$ for all $q \geqslant 1$.

Let's see the consequences of this theorem before we see the proof. Consider the complex

$$0 \to \Omega^p_M \to \Omega^{p,0}_M \xrightarrow{\overline{\partial}} \Omega^{p,1}_M \xrightarrow{\overline{\partial}} \Omega^{p,2}_M \xrightarrow{\overline{\partial}} \cdots$$

on a complex manifold M. Ω^p_M is the sheaf of holomorphic p-forms, and $\Omega^{p,q}_M$ is the sheaf of C^∞ (p,q)-forms. This complex is exact at $\Omega^{p,0}_M$, since a $(p,0)$-form α is holomorphic if and only if $\overline{\delta}\alpha = 0$. Then **??** implies that this complex is exact at $\Omega^{p,q}_M$ for $q > 0$. $\Omega^{p,q}_M$ is fine, hence acyclic, and therefore

$$\mathrm{H}^q(M,\Omega^p_M) = \mathrm{H}^q(\Gamma(M,\Omega^{p,\bullet}_M)) = \mathrm{H}^{p,q}_{\overline{\partial}}(M).$$

So now we can compare two gadgets that we don't really understand.

Similarly, we have an exact sequence,

$$0 \to \mathbb{C} \to \Omega^0_M \xrightarrow{\delta} \Omega^1_M \xrightarrow{\delta} \Omega^2_M \xrightarrow{\cdots}$$

but they are not fine – there are no partitions of unity for holomorphic functions.

Example 84. If $\dim M = 1$, then we have a short exact sequence

$$0 \to \mathbb{C} \to \Omega^0_M \to \Omega^1_M \to 0$$

This gives a long exact sequence of cohomology

$$0 \to \mathrm{H}^0(M,\mathbb{C}) \xrightarrow{f} \mathrm{H}^{0,0}(M) \xrightarrow{g} \mathrm{H}^{1,0}(M) \to \mathrm{H}^1(M,\mathbb{C}) \to \mathrm{H}^{0,1}(M) \to \mathrm{H}^{1,1}(M) \to \mathrm{H}^2(M,\mathbb{C}) \to 0$$

Assuming that M is compact and connected, then $\mathrm{H}^0(M,\mathbb{C}) = \mathbb{C}$ and also $\mathrm{H}^{0,0}(M) = \mathbb{C}$, so the map f above is an isomorphism and the map g is the zero map.

Lecture 10 — 4 February 2016

Proof of the $\overline{\partial}$-Poincaré Lemma

Today we'll prove the $\overline{\partial}$-Poincaré Lemma.

The goal is to show that $\mathrm{H}^{p,q}_{\overline{\partial}}(\Delta(r)) = 0$ for all $q > 0$. Recall that $\Delta(r) = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_i| < r \forall\, i\}$.

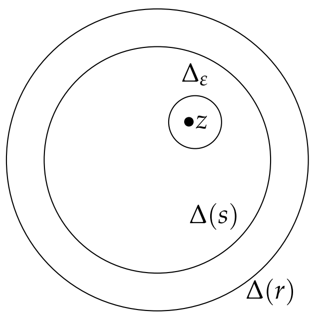
The first step is to show that if α is a (p,q)-form on $\Delta(r)$ with $\overline{\partial}\alpha = 0$. Then for any $s < r$, there exists a $(p,q-1)$-form β on $\Delta(r)$ such that $\overline{\delta}\beta = \alpha$ on $\Delta(s)$.

Step 1, Special Case

Proof of Theorem 83, special case. Let's do a special case first. If $(p,q) = (0,1)$ and $n = 1$. Let $\alpha = f(z)d\overline{z}$. Then $\overline{\partial}\alpha = 0$ is automatic. We need a C^∞ function β such that ${}^{\partial\beta}/_{\partial\overline{z}} = f(z)$. Let $\zeta, z \in \Delta(r)$ with z fixed and ζ varying. Then

$$d\left(\frac{\beta d\zeta}{\zeta - z}\right) = \frac{\partial\beta}{\partial\overline{\zeta}}\frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z}.$$

Let $z \in \Delta(s)$ for given $s < r$, Δ_ε a small disk centered at z contained in $\Delta(s)$.



Apply Stoke's Theorem to $\Delta(s)\backslash\Delta_\varepsilon$.

$$\int_{\partial\Delta(s)} \frac{\beta(\zeta)d\zeta}{\zeta - z} - \int_{\partial\Delta_\varepsilon} \frac{\beta(\zeta)d\zeta}{\zeta - z} = \int_{\Delta(s)=\Delta_\varepsilon} \frac{\partial\beta}{\partial\overline{\zeta}}\frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z}$$

The second integral on the left converges to $2\pi i\beta(z)$ as $\varepsilon \to 0$. Hence, we get the **Generalized Cauchy Integral formula:**

$$2\pi i\beta(z) = \int_{\partial\Delta(s)} \frac{\beta d\zeta}{\zeta - z} + \int_{\Delta(s)} \frac{\partial\beta}{\partial\overline{\zeta}} \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z}$$

Henceforth write ${}^{\partial\beta}/_{\partial\overline{\zeta}}$ as $\beta_{\overline{\zeta}}$.

Taking complex conjugates and replacing β by $\overline{\beta}$,

$$-2\pi i\beta(z) = \int_{\partial\Delta(s)} \frac{\beta d\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \int_{\Delta(s)} \beta_{\zeta} \frac{d\zeta \wedge d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \tag{4}$$

Note that if $\beta_{\overline{\zeta}} = f(\zeta)$, we get

$$2\pi i\beta(z) = \int_{\Delta(s)} f(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} + g(z) \tag{5}$$

where g is a holomorphic function in z.

We want to check that (5) indeed defines the desired β, noting that $g(z)$, being holomorphic, doesn't affect ${}^{\partial\beta}/_{\partial\overline{\zeta}} = f$. Note that

$$d(f(\zeta)\log|\zeta - z|^2 d\overline{\zeta}) = f_{\zeta} \log|\zeta - z|^2 d\zeta \wedge d\overline{\zeta} + \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

Apply Stoke's theorem to $\Delta(s) - \Delta_{\varepsilon}$ again. As $\varepsilon \to 0$,

$$\int_{\partial\Delta_{\varepsilon}} f(\rho)\log|\rho - z|^2 d\overline{\zeta} \longrightarrow 0$$

since if $|f(\zeta)| \leqslant B$ on $\Delta(s)$, then

$$\left| \int_{\partial\Delta_{\varepsilon}} f(\rho)\log|\rho - z|^2 d\overline{\zeta} \right| \leqslant 2\pi\varepsilon \cdot 2 \cdot B \log\varepsilon \xrightarrow[\varepsilon \to 0]{} 0.$$

Thus, we get

$$\int_{\partial\Delta(s)} f(\zeta)\log|\zeta - z|^2 d\overline{\zeta} - \int_{\delta(s)} f_{\zeta} \log|\zeta - z|^2 d\zeta \wedge d\overline{\zeta} = \int_{\partial\Delta(s)} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} = 2\pi i\beta(z)$$

Notice that this is equal to (5). Differentiating under the integral sign with respect to $\overline{z}$, we get

$$-\int_{\partial\Delta(s)} \frac{f(\zeta)}{\overline{\zeta} - \overline{z}} d\overline{\zeta} + \int_{\Delta(s)} \frac{f_{\zeta}}{\overline{\zeta} - \overline{z}} d\zeta \wedge d\overline{\zeta} = 2\pi i \frac{\partial\beta}{\partial\overline{\zeta}}$$

Compare this to (4) to see that the above is equal to $2\pi i f(z)$. Therefore, this particular choice of β solves the equation we're trying to solve. □

Step 1, General Case

Proof of Theorem 83, general case. Proof by induction on n. Let n be arbitrary, (p,q) arbitrary with $q \neq 0$. Let H_j be the induction hypothesis that the claim is true if α does not involve $d\overline{z}_{j+1},\dots,d\overline{z}_n$.

The base case H_0 is that $\alpha = 0$, so choose $\beta = 0$.

Now we want to show H_n. Assume H_{j-1} for some j, and suppose α doesn't involve $d\overline{z}_{j+1},\dots,\partial\overline{z}_n$. We can write $\alpha = \mu_1 + d\overline{z}_j \wedge \lambda_i$, where μ_1, λ_1 don't involve $d\overline{z}_j,\dots,d\overline{z}_n$. We see that λ_1 is type $(p,q-1)$ and μ is type (p,q). Since $\overline{\partial}\alpha = 0$, the coefficients of μ_1 must be holomorphic in the variables $z_{j+1},\dots,z_n$.

By the special case applied to the variable z_j, we can find some λ_1' of type $(p,q-1)$ such that

$$\frac{\partial\lambda_1'}{\partial\overline{z}_j} = \lambda_1,$$

and the coefficients of λ_1' are still holomorphic in $z_{j+1},\dots,z_n$. Then

$$\overline{\partial}\lambda_1' - d\overline{z}_j \wedge \lambda_1 = \nu$$

doesn't contain $d\overline{z}_j,\dots,d\overline{z}_n$. And by substituting,

$$\alpha = \mu_1 - \nu + \overline{\partial}\lambda_1'$$

and $\overline{\partial}\alpha = 0$ implies that $\overline{\partial}(\mu_1 - \nu) = 0$.

So we can apply the inductive hypothesis H_{j-1}, which implies that there exists a $(p,q-1)$-form ρ on $\Delta(r)$ satisfying $\mu_1 - \nu = \overline{\partial}\rho$ on $\Delta(s)$. Then $\alpha = \overline{\partial}(\lambda_1' + \rho)$ on $\Delta(s)$. This concludes the general case. □

Step 2

Now that we've done the first step, the next is to do the full $\overline{\partial}$-Poincaré Lemma. We need a sequence $r_k \to r$, with $r_k < r$, and a sequence β_k of solutions to $\overline{\partial}\beta_k = \alpha$ on $\Delta(s)$ with the β_k converging uniformly to some β.

Fix a monotone increasing sequence $r_k \to r$. Fix a k and assume $q \geqslant 2$. Let rho_k be a C^∞ bump function with support of $\Delta(r_{k+1})$ and identically 1 on $\Delta(r_k)$.