Topological Hochschild homology

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§ 0 Introduction

In this paper we construct a topological version of Hochschild homology.

One motivation for doing this, is the relation of K-theory to ordinary Hochschild homology.

Recall that the Hochschild homology of a bimodule M over a ring R can be described as the simplicial group $HH(R)$

$$[i] \mapsto M \otimes R^{\otimes i}$$

$$d_j(m \otimes r_1 \otimes \dots r_i) = \begin{cases} mr_1 \otimes r_2 \otimes \dots r_i & j = 0 \\ m \otimes \dots \otimes r_j r_{j+1} \otimes \dots r_i & 0 < j < i \\ r_i m \otimes r_1 \otimes \dots \otimes r_{i-1} & j = i \end{cases}$$

$$s_j(m \otimes r_1 \otimes \dots r_i) = m \otimes r_1 \otimes \dots r_{j-1} \otimes 1 \otimes r_j \otimes \dots r_i$$

This functor satisfies Morita equivalence, i.e. if $M_n(R)$ is the matrix ring of R, then

$$HH(M_n(R), M_n(R)) \simeq HH(R,R) .$$

Using Morita equivalence we can define a map from K-theory

$$i : K(R) \longrightarrow THH(R)$$

Since there is an inclusion of simplicial objects [2]

$$BGL_n(R) \longrightarrow HH(M_n(R), M_n(R)) .$$

In his work on $A(X)$, Waldhausen discovered a similar map [9]. The natural generality of his construction cf. [3], is a map defined for a simplicial group G

$$i_G : A(BG) \longrightarrow Q(\Lambda BG_+)$$

where ΛBG denotes the free loop space. The precise definition of this map is somewhat complicated.

In order to see that these maps are similar, one needs a general framework. It is possible to consider the "groupring" $Q[G_+]$ and a ring R as special cases of "rings up to homotopy" [4], [7].

Goodwillie noted, that in order to define the Hochschild homology of a ring up to homotopy, one would have to replace the simplicial group above with a simplicial spectrum, and the tensor products occuring with smash products of spectra. If we do this with the ring $\mathbb{Z}$ considered as a ring up to homotopy we obtain a new object, the topological Hochschild homology of $\mathbb{Z}$.

In this paper the topological Hochschild homology is constructed for a large class of rings up to homotopy. It is shown that the map from K-theory is a map of rings up to homotopy. Finally, it is shown that THH(R) is a cyclic object in the sense of [1], [3], and that the map from K-theory to topological Hochschild homology is - in a weak sense - the inclusion of a trivial cyclic object.

I want to thank F. Waldhausen for discussions on this subject. Several of the crucial ideas in the paper are due to him.

§ 1

We want to define Hochschild homology of a ringspectrum R . This should be a spectrum, and in case R is commutative, the Hochschild homology should be a commutative ringspectrum in itself.

The first attempt is to define Hochschild homology as a simplical spectrum, which in degree i is the smash product of $i+1$ copies of R . This is ok in the homotopy category. We obtain a simplical object, which we want to lift to a simplicial spectrum.

One problem is that $R \wedge R$ is not a spectrum but a bispectrum. We have to find a way of associating a spectrum to this bispectrum, so that we have a map $R \wedge R \to R$ representing the product, and such that we can use this product to construct the simplicial object in question.

It seems reasonable to assume that one can do the construction for rings up to homotopy in the sense of [6], [7]. I will limit myself to a more restricted kind of ring.

One special type of ring up to homotopy is a monad [4] . The monads are functors F with two structure maps

$$1_X : X \longrightarrow F(X)$$

$$\mu_X : FF(X) \to F(X)$$

satisfying appropriate associativity and naturality conditions. We consider monads which are functors from simplicial sets to simplicial sets.

We will be concerned with functors satisfying somewhat conditions.

<u>Definition 1.1.</u> A functor with smash product (FSP) is a functor F from finite pointed simplicial sets to pointed simplicial sets, together with two natural transformations

$$1_X : X \longrightarrow F(X)$$

$$\mu_{X,Y} : F(X) \wedge F(Y) \longrightarrow F(X \wedge Y)$$

such that $\mu(\mu \wedge id) = \mu(id \wedge \mu)$, $\mu(1_X \wedge 1_Y) = 1_{X \wedge Y}$, and such that the limit system

$$\pi_r(\Omega^i F(S^i X)) \to \pi_r(\Omega^{i+1} F(S^{i+1} X))$$

given by product with $1_{S^1} : S^1 \to F(S^1)$ stabilizes for every r .

We say that F is commutative if the following diagram commutes

$$\begin{array}{ccc} F(X) \wedge F(Y) & \xrightarrow{\mu} & F(X \wedge Y) \\ \downarrow & & \downarrow \\ F(Y) \wedge F(X) & \xrightarrow{\mu} & F(Y \wedge X) \end{array} .$$

Remark: If F is a monad with a certain extra condition, then it is also an FSP . The condition is that given a simplicial set X , we can assemble the maps associated to the simplices in X to a simplicial map

$$X \wedge F(Y) \longrightarrow F(X \wedge Y) .$$

Then we can define $\mu_{X,Y}$ as the composite

$$F(X) \wedge F(Y) \to F(X \wedge F(Y)) \to FF(X \wedge Y) \to F(X \wedge Y) .$$

Now consider the infinite loopspaces

$$\lim_{\vec{n}} \Omega^n F(S^n) \quad \text{and} \quad \lim_{n,m} \Omega^{n+m}(F(S^n) \wedge F(S^m)) .$$

We would like to construct an infinite loop map representing the product

$$F(S^n) \wedge F(S^m) \to F(S^{n+m}) .$$

What we can do, is to construct a map

$$\lim_{n,m} \Omega^{n+m}(F(S^n) \wedge F(S^m)) \to \lim_{m,n} \Omega^{n+m} F(S^n \wedge S^m) .$$

The right hand side is isomorphic to $\lim_{m} \Omega^m F(S^m)$, but not equal. It is not clear that we can choose this equivalence so that we obtain a simplicial infinite loopspace with

$$[0] \longmapsto \lim_{n} \Omega^{n} F(S^{n})$$

$$[1] \longmapsto \lim_{m,n} \Omega^{m+n} F(S^{m}) \wedge F(S^{n}) \quad .$$

We will avoid this difficulty by constructing a different limit.

Let X be a finite set. Let S^X denote the sphere which we obtain by taking the smash product of copies of S^1 indexed by X. Using this sphere, we can define loopspace and suspension functors $\Omega^X(-)$ and $\Sigma^X(-)$.

For an FSP F, we obtain a functor $\Omega^X F(S^X)$ from the category of finite sets and isomorphisms to the category of simplicial sets and homotopy-equivalences.

The stabilization map

$$S^X \wedge F(-) \to F(S^X) \wedge F(-) \to F(S^X \wedge -)$$

allows us to extend this functor to a functor on finite ordered sets and order preserving injective maps.

In case F is commutative, we can extend the functor to a functor on finite sets and injective maps.

Let I be one of these last two categories. We consider the limits

$$\lim_{(X,Y)\in I^2} \Omega^{X \amalg Y} F(S^X) \wedge F(S^Y)$$

$$\lim_{X \in I} \Omega^X F(S^X) \quad .$$

The product map $I^2 \to I$ given by μ is covered by a map of limit systems using the product

$$\mu : F(S^X) \wedge F(S^Y) \to F(S^{X \amalg Y}) \quad .$$

The trouble is, that the index category is not filtering. In particular, we do not know that the limit has the correct homotopy type.

However, there is a homotopy version of the limit, which has the correct properties.

Let C be a category, $F : C \to$ simplicial sets a functor. We define $L_C F$ to be the bisimplicial set

$$[i] \mapsto \coprod_{(f_1,\ldots,f_i)} F(\text{source } f_1)$$

where $(f_1,\ldots,f_i)$ runs over composable morphisms $X_o \xrightarrow{f_1} X_1 \to \ldots \xrightarrow{f_i} X_i$ in C.

The structure maps of $L_C F$ are so defined, that there is a simplicial map $L_C F \to BC$.

The degeneracies are given by introducing the identical map in the index set, and

$$d_o : \coprod_{(f_1,\ldots,f_i)} F(\text{source } f_1) \to \coprod_{(f_2,\ldots,f_i)} F(\text{source } f_2)$$

is given by applying

$$F(f_1) : F(\text{source } f_1) \to F(\text{source } f_2) \quad .$$

We will need this construction not just for the category of ordered finite sets, but also for the category of finite sets and injective maps, and also for products of these with themselves.

We claim, following Illusie [5] that in these cases the homotopy limit behaves like a limit.

It is convenient to introduce an abstract notion.

Definition 1.2. A category C is a good limit category if it has the following properties.

1^o There is an associative product

$$\mu : C \times C \to C \quad .$$

2^o There are natural transformations between

that way

$$\mu : C \times C \to C \quad \text{and}$$
$$pr_i : C \times C \to C \qquad i = 1, 2 \quad .$$

3^o There is a filtration

$$C = F_o C \supset F_1 C \supset \ldots \quad .$$ non-empty

such that $\mu(F_i C, F_j C) \subset F_{i+j} C$. [consistent with (2)] ?

Example 1.3.1. The category I of finite ordered sets and ordered injective maps is a good index category. μ is given by concatenation, and $F_i I$ is the full subcategory of objects with cardinality greater or equal to i.

Example 1.3.2. The category I^{comm} of finite sets and injective maps is also a good index category.

This category has an extra structure, a natural transformation between $\mu(A,B)$ and $\mu(B,A)$.

If C is a good index category, so is C^n. Iteration of μ defines a functor $\bar{\mu} : C^n \to C$. A natural transformation from $F \circ \bar{\mu}$ to G defines maps

$$L_{C^n}(G) \to L_{C^n}(F \circ \bar{\mu}) \to L_C(F) \quad .$$

Lemma 1.4. If C is a good index category, $G : C \to$ simplicial sets a functor, then the inclusion $F_i C \subset C$ induces a homotopy equivalence

$$L_{F_i C} G \to L_C G \quad .$$

Proof. Let $A \in F_i C$. There is a functor $A * : C \to C$ given by $A*(B) = \mu(A,B)$. This functor raises filtration by i, so in particular it factors over $F_i(C) \to C$.

This functor induces a map

$$A * : L_C(G) \to L_C(G) \quad ,$$

since there is a natural transformation between $A*$ and the identity on $L_C(G)$, this map is homotopic to the identity.

By the same argument, the restriction of $A*$ to $L_{F_i(C)} G \to L_{F_i(C)} G$ is also homotopic to the identity. By the homotopy extension property, we can extend this homotopy to a homotopy of

$$A * : L_C(G) \to L_{F_i C}(G)$$

to a map, which is a retraction

$$L_C(G) \to L_{F_i C}(G) \quad .$$

Composing homotopies, we see that $L_{F_iC}(G)$ is a deformation retract of $L_C(G)$.

Now, let C be a good index category, $G : C \to$ simplicial sets a functor with the property that

$$G(f) : G(X) \to G(X')$$

is a $\lambda(i)$-equivalence for $f : X \to X' \subset F_i(C)$.

Theorem 1.5. Let $X \in F_i(C)$. The inclusion of the category consisting of the identity of X in C defines a $\lambda(i)$-equivalence

$$G(X) \to L_C(G) .$$

Proof. By lemma 1.4, it suffices to prove that $G(X) \to L_{F_i(C)}(G)$ is a $\lambda(i)$-equivalence. Replacing G by a coskeleton of G , we see that it suffices to prove that if all maps $G(f) : G(X) \to G(Y)$ are homotopy equivalences, then

$$G(X) \to L_{F_iC}(G)$$

is also a homotopy equivalence.

This statement is equivalent to the statement that $B(F_iC)$ is contractible, since we have a fibration

$$L_{F_iC}(G) \to BF_i(C)$$

with fibre homotopy equivalent to $G(X)$.

But the product μ induces an H-space structure on BF_iC , and by conditions 2^o and 3^o it is connected. A well known trick, using the homotopy equivalence

$$BF_iC \times BF_iC \to BF_iC \times BF_iC$$

$$(a,b) \mapsto (ab,b)$$

shows that BF_iC has a homotopy inverse .

By condition 2^o , the following maps are homotopic

$$pr_1, \mu : BF_iC \times BF_iC \to BF_iC .$$

Composing with the skew diagonal

$$BF_iC \xrightarrow[(*,I*)]{(1,-1)} BF_iC \times BF_iC$$

we obtain that BF_iC is contractible.

Now, let F be an FSP . As above, we obtain a functor

$$X \to \Omega^X F S^X$$

defined on I or I^{comm} , according to whether F is commutative or not.

Definition 1.6. $(F^i)^s$ is the simplicial object

$$\lim_{I^{i+1}} (\Omega^{X_0 \amalg \ldots X_i} F(S^{X_0}) \wedge F(S^{X_2}) \wedge .. F(S^{X_i}))$$

$(F^i)^s_{comm}$ is defined for commutative F as the corresponding limit over $(I^{comm})^{i+1}$.

Products define maps

$$(F^i)^s \to (F^j)^s$$

corresponding to all maps $[1,\ldots,i] \to [1,\ldots,j]$ which preserve the cyclic ordering. In particular, we can define the topological Hochschild homology of F as the simplicial object

$$THH(F) :$$
$$[i] \mapsto (F^i)^s$$

with the usual simplicial structure maps.

If F is commutative, we can replace $(F^i)^s$ with $(F^i)^s_{comm}$, since the map induced by inclusion $I \to I^{comm}$

$$(F^i)^s \to (F^i)^s_{comm}$$

is a homotopy equivalence.

§ 2

In this paragraph we will examine different structures in THH(F). In particular, we will show that if F is commutative, THH(F) is a ring up to homotopy, with a cyclic structure compatible with this ring structure.

Finally, we will construct K-theory of F and show that K(F) maps to THH(F) respectively maps as a ring up to homotopy, when F is commutative.

We are going to use the theory of hyper-Γ-spaces [11].

Re call that a Γ-space [8] is given by a functor from the category Γ^{op} of finite sets to the category of spaces.

Given a Γ-space

$$F : \text{Finite sets} \to \text{spaces} ,$$

we can construct an infinite loop space. If the Γ-space has the additional property, that the component shift maps

$$F(X) \to F(X) \times F(Y) \overset{?}{\to} F(X \amalg Y)$$

are homotopy equivalences for each X , then this infinite loop-space is homotopy equivalent to $\mathbb{Z} \times F(\text{point})$.

Similarly, a hyper-Γ-space is a functor

$$F : \Gamma^{op} \wr \Gamma^{op} \to \text{spaces}$$

from the category of finite sets of finite sets.

Again we can construct an infinite loop space. In this case we have a product on this space, making it into a ring up to homotopy [11].

In order to construct Γ-spaces and hyper-Γ-spaces, it suffices to construct functors from the isomorphisms in Γ^{op} respectively $\Gamma^{op} \wr \Gamma^{op}$; with certain additional properties.

Let $G : S \to$ spaces be a functor from the category of finite sets and isomorphisms , with an additional natural transformation

$$\alpha : G(X) \times G(Y) \to G(X \amalg Y)$$

satisfying commutativity.

Given such a G , we can construct a Γ-space X_G as

$$X_G(C) = \coprod_{\{A_{c_1},\ldots,A_{c_n}\}} G(A_{c_1}) \times \ldots \times G(A_{c_n})$$

where A_{c_i} are finite sets, indexed by the elements $c_1,\ldots,c_n$ of C .

For instance, if G is the functor which to a finite set associates the classifying space of its automorphisms, then

$$X_G(C) \simeq \coprod_{f:C\to N} B\Sigma(f(c_1)) \times B\Sigma(f(c_2)) \times \ldots B\Sigma(f(c_n))$$

and the associated Γ-space is the group completion of

$$\coprod_{i\in N} B\,\Sigma(i)\ .$$

If G has the property that the component shift maps, given by points in $G(Y)$

$$G(X) \to G(X) \times G(Y) \overset{\alpha}{\to} G(X \amalg Y)$$

is a homotopy equivalence for each X, then the Γ-space is homotopy equivalent to $\mathbb{Z} \times G(\text{point})$.

In the same way, if we have a product

$$\mu : G(X) \times G(Y) \to G(X \times Y)$$

which is commutative and distributive over α, then X_F can be extended to a hyper-Γ-space.

We now define the K-theory of an FSP, and show that it is a Γ-space, respectively a hyper-Γ-space in the commutative case using the theory above. First, we note that we can form the matrix - FSP :

Definition 2.1. Let F be an FSP, and A a finite set. Then

$$(M_A F)(X) = \mathrm{Map}^o(A_+, A_+ \wedge F(X))$$

The functor $M_A F(-)$ is an FSP in a natural way. $1_{M_A F}$ is the adjoint of $\mathrm{id} \wedge 1 : A_+ \wedge X \to A_+ \wedge F(X)$ and $\mu_{X,Y}$ is the composite

$$\mathrm{Map}^o(A_+, A_+ \wedge F(X)) \wedge \mathrm{Map}^o(A_+, A_+ \wedge F(Y)) \to \mathrm{Map}^o(A_+, A_+ \wedge F(X) \wedge F(Y)) \to \mathrm{Map}^o(A_+, A_+ \wedge F(X \wedge Y))$$

where the first map is given by

$$(f,g) \to (f \wedge \mathrm{id}_{F(Y)}) \circ g$$

and the second is induced by $\mu_{X,Y}$.

Let F be an FSP. A homotopy unit of F is a map $f : S^X \to F(S^X)$, having a homotopy inverse $g : S^Y \to F(S^Y)$, that is

$$S^X \wedge S^Y \xrightarrow{f\wedge g} F(S^X) \wedge F(S^Y) \overset{\mu}{\to} F(S^X \wedge S^Y)$$

is homotopic to $1_{S^X \wedge S^Y}$.

Definition 2.2. The monoid of homotopy units of F is defined as

$$F^* = L_I(\text{homotopy units in } \Omega^X F(S^X)) .$$

We can now for any finite set A consider $GL_A(F)$, the monoid of homotopy units of $M_A(F)$.

There are maps

$$\alpha : GL_A(F) \times GL_B(F) \to GL_{A \amalg B}(F)$$

$$\mu : GL_A(F) \times GL_B(F) \to GL_{A \times B}(F)$$

induced by sum resp. product of maps.

α is given by the product $(f,g) \mapsto h$ where f,g,h are adjoint respectively, to

$$\bar{f} : A_+ \wedge S^X \to A_+ \wedge F(S^X)$$

$$\bar{g} : B_+ \wedge S^Y \to B_+ \wedge F(S^Y)$$

$$\bar{h} = (id \wedge \mu_{X,Y})(\bar{f} \wedge 1_{S^Y}) \vee (id \wedge \mu_{X,Y})(1_{S^X} \wedge \bar{g})$$

μ is defined in the case where F is commutative, and induced by the product

$$(A_+ \wedge F(X)) \wedge (B_+ \wedge F(Y)) \to (A \times B)_+ \wedge F(X \wedge Y) .$$

In order that this product induces a product $GL_A \times GL_B \to GL_{A \times B}$, we have to modify the definition of GL_A slightly. We replace L_I(homotopy units of $M_A F$) by the commutative version

$$L_{I\text{comm}}(\text{homotopy units of } M_A F) .$$

These structure maps are commutative, and μ is distributive over α .

Out of these monoids, we can construct two (hyper-)Γ-spaces.

Definition 2.3. $K(F)$ is the hyper-Γ-space given by the functor

$$G(A) = BGL_A(F)$$

defined on finite sets and isomorphisms, ordered, respectively unordered as to whether F is commutative, or not.

The structure maps are the classifying maps $B\alpha$ and $B\mu$.

Definition 2.4. $N^{cy}(F)$ is the (hyper-)Γ-space given by the functor

$$G(A) = N^{cy}(GL_A(F), GL_A(F))$$

the cyclic bar construction [9] of $GL_A(F)$ acting on itself.

Using the same method, we can make THH(F) into a Γ-space. There is a map

$$M_A(F)(X) \wedge M_B(F)(Y) \to M_{A \amalg B}(F)(X \wedge Y)$$

analogous to α above. In case F is commutative, we use the model constructed using I^{comm} instead of I . Then we can use the product on F to construct a map

$$\mu : THH(M_A(F)) \times THH(M_B(F)) \to THH(M_{A \times B}(F))$$

which is commutative, and distributive over α .

Lemma 2.5. The Γ-space $THH(M_A(F))$ is homotopy equivalent to THH(F) .

Proof. It suffices to show that the inclusions

$$THH(F) \to THH(M_A(F))$$

are equivalences for each nonempty A . This is a version of Morita equivalence. We follow the argument of [9].

Let V_A and H_A be the FSP $A_+ \wedge F(-)$ respectively $Map^o(A_+, F(-))$. There a pairings, representing actions of F and $M_A(F)$ on these, e.g.

$$V_A(P) \wedge M_A F(Q) \to V_A(P \wedge Q) \quad .$$

We can form the bisimplical object

$$[i,j] \to L_{I^{i+1} \times I^{j+1}} \Omega^{X_0 \amalg \dots X_i \amalg Y_0 \amalg \dots Y_j} V_A(S^{X_0}) \wedge M_A F(S^{X_1}) \wedge \dots \wedge M_A F(S^{X_i}) \wedge H_A(S^{Y_0}) \wedge F(A^{Y_1}) \wedge \dots F(S^{Y_j}) \quad .$$

We claim that the two multiplication maps

$$V_A(S^{X_0}) \wedge M_A F(S^{X_1}) \wedge \dots \quad H_A(S^{Y_0}) \to F(S^{X_0 \amalg \dots Y_0})$$

$$H_A(S^{Y_0}) \wedge F(S^{Y_1}) \wedge \dots \quad V_A(S^{X_0}) \to M_A F(S^{Y_0 \amalg \dots X_0})$$

induce homotopy equivalences to $THH(M_A F)$ respectively THH(F) .

This is equivalent to the statement that the objects

$$[i] \mapsto L_{I^{i+2}} \, \Omega^{X_o \amalg \dots X_{i+1}} \, V_A(S^{X_o}) \wedge F(S^{X_1}) \wedge \dots H_A(S^{X_{i+1}})$$

$$[i] \mapsto L_{J^{i+2}} \, \Omega^{X_o \amalg \dots X_{i+1}} \, H_A(S^{X_o}) \wedge M_A F(S^{X_1}) \wedge \dots V_A(S^{X_{i+1}})$$

are equivalent to $L_I \, \Omega^X \, M_A(S^X)$ respectively $L_I \, \Omega^X \, F(S^X)$.

Now, consider the special case A=point. We can replace the limit by the corresponding limit over the subcategory of I^{i+2} consisting of tuples of sets with more then N elements.

Then the multiplication map in degree [i] can be described as

$\Omega^{X_0 \amalg \dots X_{i+1}}$ applied to the product

$$F(S^{X_0}) \wedge \dots F(S^{X_{i+1}}) \to F(S^{X_0 \amalg \dots X_{i+1}})$$

It follows that multiplication induces a homotopy equivalence.

The general case reduces to this, by rewriting for instance

$$H_A(S^{X_o}) \wedge M_A(S^{X_1}) \wedge \dots M_A(S^{X_i}) \wedge V_A(S^{X_{i+1}})$$

as a subspace of

$$A_+^{\wedge [i]} \wedge \mathrm{Hom}(A_+^{\wedge [i]}, F(S^{X_o}) \wedge \dots F(S^{X_{i+1}}))$$

and noting that the inclusion of this subspace is highly connected.

The Γ-space THH(F) has a cyclic structure in the sense of [3] . If F is commutative, then the hyper-Γ-space THH(F) has a cyclic structure.

The same statements are true for the (hyper-)Γ-space $N^{cy}(F)$.

The inclusion

$$GL_A(F) \to L_I \, \Omega^X \, M_A(F)(S^X)$$

defines a map of (hyper-)Γ-spaces compatible with the cyclic structure:

$$N^{cy}(F) \to THH(F) \quad .$$

We finally examine the relation between K(F) and $N^{cy}(F)$.

Let X be a monoid, $N^{cy}(X,X)$ the cyclic barconstruction of X acting on itself. This is again a cyclic object.

If X is a group, we can include BX in $N^{cy}(X,X)$ as the cyclic subobject consisting in degree n of

$$\{(x_o,\dots,x_n)\in(N^{cy})_n \quad | x_o x_1 \dots x_n = 1\} \quad .$$

The isomorphism to the standard barconstruction is given by projection onto the last n coordinates.

This object is not equivalent to the constant cyclic object BX. In order to relate $N^{cy}(X,X)$ to this constant object, we replace X by an equivalent category.

In fact, for any category $\mathcal{C}$, we can define $N^{cy}(\mathcal{C},\mathcal{C})$ as the simplicial set

$$[n] \mapsto \left\{(f_o,\dots,f_n)\in(\text{Morph }\mathcal{C})^{n+1};\ \begin{matrix} f_i \text{ and } f_{i+1} \text{ composable} \\ f_n \text{ and } f_o \text{ composable} \end{matrix}\right\} \quad .$$

It is clear that a functor $\mathcal{C}\to\mathcal{D}$ induces a map

$$N^{cy}(\mathcal{C},\mathcal{C}) \to N^{cy}(\mathcal{D},\mathcal{D}) \quad .$$

It is not true that a natural transformation induces a homotopy.

Remark 2.6. If $F,G:\mathcal{C}\to\mathcal{D}$ are naturally equivalent through isomorphisms, then the induced maps are homotopic after realization.

To see this, consider the category I with two object and exactly one isomorphism between them. The natural equivalence defines a functor

$$I \times \mathcal{C} \to \mathcal{D}$$

inducing

$$N^{cy}(I\times\mathcal{C},I\times\mathcal{C}) \to N^{cy}(\mathcal{D}\times\mathcal{D}) \quad .$$

The simplicial set $N^{cy}(I\times\mathcal{C},I\times\mathcal{C})$ is isomorphic to the diagonal of the bisimplicial set

$$N^{cy}(I,I) \times N^{cy}(\mathcal{C},\mathcal{C}) \quad .$$

A path in $|N^{cy}(I,I)|$ between its two zero simplices defines a homotopy between $N^{cy}(F)$ and $N^{cy}(G)$.

This path is given by

$$(f,f^{-1}) \in N^{cy}(I,I)_1 \quad .$$

In particular, the remark shows that equivalent categories have homotopy equivalent realizations $|N^{\overset{C}{Y}}(C,C)|$.

The application is to a certain bicategory.

Let $Sq(C)$ be the bicategory consisting of commutative diagrams in C

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ Z & \to & T \end{array}$$

[10] . There are two ways of forming a nerve in this bicategory, yielding two different (but abstractly isomorphic) simplicial categories. Let us denote them by $Nerve_1(Sq(C))$ and $Nerve_2(Sq(C))$.

There is an inclusion of C in the simplicial category $Nerve_1\ Sq(C)$.

In case all morphisms in C are isomorphisms, this inclusion is an equivalence of categories in each degree.

There is also an inclusion of the simplicial set BC as the objects of the simplicial category $Nerve_1\ Sq(C)$.

Lemma 2.7. If all morphisms of C are isomorphisms, then the two maps

$$Nerve(C) \to Nerve_1(Sq(C)) \to Nerve_1\ Nerve_2(Sq(C))$$

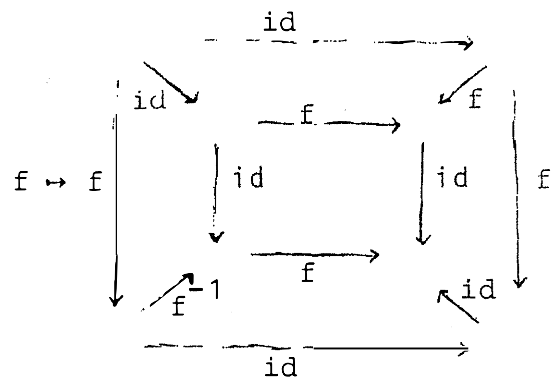
$$Nerve(C) \to Nerve_2(Sq(C)) \to Nerve_2\ Nerve_1(Sq(C))$$

are homotopic.

Proof. We can explicitely write down a binatural transformation between the functors

$$f \mapsto \begin{array}{ccc} \cdot & \xrightarrow{id} & \cdot \\ f\downarrow & & \downarrow f \\ \cdot & \xrightarrow[id]{} & \cdot \end{array} \qquad f \mapsto \begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ id\downarrow & & \downarrow id \\ \cdot & \xrightarrow[f]{} & \cdot \end{array}$$

as follows:



We can now for any monoid X consider the diagram

$X = \Omega B GL_A(F)$

$$\begin{array}{ccc} & & BX \\ & & \downarrow i \\ N^{cy}(X,X) & \xrightarrow{f} & N^{cy}(\mathrm{Nerve}_1 Sq(X), \mathrm{Nerve}_1 Sq(X)) \end{array}$$

where i is the inclusion of the objects. Then i is a map of cyclic objects, where BX has the trivial cyclic structure.

In case X is a group, f is a homotopy equivalence. It follows, that f is a homotopy equivalence when X is equivalent as a monoid to a group. This again is equivalent to the statement that $\pi_0(X)$ is a group.

In order to apply this to the Γ-spaces above, we note that the construction $C \mapsto \mathrm{Nerve}_1 Sq(S)$ commutes with products, so we can form the (hyper-)-Γ-space $N^{cy}(\mathrm{Nerve}_1 Sq(F))$.

There is a diagram of Γ-spaces and cyclic maps

$$\begin{array}{ccc} & & K(F) \\ & & \downarrow i \\ N^{cy}(F) & \xrightarrow{f} & N^{cy}(\mathrm{Nerve}_1 Sq(F)) \\ \downarrow & & \\ THH(F) & & \end{array}$$

Since $\pi_0 GL_A(F) = \lim_n \pi_n F(S^n)$ is a group, f is a homotopy equivalence.

§ 3

In this paragraph, we will make miscellanous remarks on the constructions we made.

In particular, we will use the cyclic structure to define a map

$$S^1 \times \lim_{\vec{n}} \Omega^n F(S^n) \to THH(F) \quad .$$

This map will be essential in a later paper, where we compute the homotopy type of $THH(F)$, in certain cases.

The first remark is that in some cases our constructions agree with known concepts.

Example 3.1. Let $F(U) = U$. Then

$$K(F) = A(*) \quad .$$

See [4]. Slightly more generally, let X be a simplicial set, and $G(X)$ the loopgroup of X , then the functor

$$F(U) = U \wedge G(X)_+$$

is a FSP satisfying the stability condition. We have

$$K(F) = A(X) \quad .$$

The topological Hochschild homology is given by the simplicial object

$$[i] \mapsto L_{I^{i+1}} \Omega^{X_0 \amalg \ldots X_i} \Sigma^{X_0 \amalg \ldots X_i} (G(X)^{i+1}{}_+)$$

with the usual structure maps, modelled on Hochschild homology.

This can be rewritten as the diagonal of a bisimplicial set whose realization in one simplicial direction is

$$[i] \mapsto L_{I^{i+1}} \Omega^{X_0 \amalg \ldots X_i} \Sigma^{X_0 \amalg \ldots X_i} (\Lambda X_+) \quad .$$

All structure maps in this object are homotopy equivalences, so

$$THH(F) \simeq \lim_{n} \Omega^n S^n(\Lambda X_+) = Q(\Lambda X_+) \quad .$$

There is a map

$$A(X) \to Q(\Lambda X_+)$$

defined as in [3].

I claim that this map agrees with the map

$$K(F) \to THH(F) \quad .$$

This follows from the fact that lemma 2.7 provides a homotopy commutative diagram

$$\begin{array}{ccc} K(F) & \longrightarrow & N^{cy}(\mathrm{Nerve}_1 \, SqF) \\ & & \uparrow \\ & & N^{cy}(F) \end{array}$$

where $K(F) \to N^{cy}(F)$ is given by the inclusion

$$BGL_A(F) \to \Lambda \; BGL_A(F)$$

followed by the identification

$$\Lambda \; BGL_A(F) \simeq N^{cy}(F) \quad .$$

It is not difficult to check that this map followed by

$$N^{cy}(F) \to THH(F)$$

agrees with the map of Waldhausen.

It is possible to define a stable version of $K(F)$.

Definition 3.2.

$K^S(F)$ is the limit of $\Omega^m X_m$, where X_m is the Γ-space $A \mapsto \mathrm{Fibre}(N^{cy}(GL_A(F), M_A F(S^m)) \to BGL_A(F))$.

The map $K(F) \to THH(F)$ factors over $K^S(F)$.

Finally, consider the space

$$F = L_I \; \Omega^X \; F(S^X) \quad .$$

Then F is a (hyper)-Γ-space. We say that F is the underlying Γ-space of the FSP $F(-)$.

We can define a map

$$\lambda : S^1 \times F \to THH(F) \quad .$$

There is a simplicial model of S^1 as

$$[i] \mapsto \{0,1,\ldots,i\}$$

with the usual cyclic structure maps, e.g.

$$d_k(j) = \begin{cases} j & j \le k \\ j-1 & j > k \end{cases} \qquad k < i$$

$$d_i(j) = \begin{cases} j & j < i \\ 0 & j = k \end{cases}$$

We can define λ in degree i as

$$\coprod_{j=0}^{i} L_I \, \Omega^X F(S^X) \to L_{I^{i+1}} (\Omega^{X_0 \amalg \ldots X_i} (F(S^{X_0}) \wedge \ldots \wedge F(S^{X_i}))$$

by including the k^{th} summand in the k^{th} factor.

By smashing with the identity on A, we obtain maps

$$S^1 \times F \to S^1 \times M_A F \to THH(M_A(F)) \quad .$$

These maps combine to a map of (hyper)-Γ-spaces parametrized by S^1.

We note that the composite

$$S^1 \times GL_1(F) \to S^1 \times F \xrightarrow{\lambda} THH(F)$$

factors over the map

$$S^1 \times GL_1(F) \to N^{cy}(F^*,F^*)$$

given in degree i by sending the k^{th} summand in

$\coprod_{k=0}^{i} L_I$(homotopy units in $\Omega^X F(S^X)$) to the factor in

$\prod_{k=0}^{i} L_I$(homotopy units in $\Omega^X F(S^X)$) .

In other words, we have a commutative diagram

$$\begin{array}{ccc} S^1 \times F^* & \longrightarrow & S^1 \times F \\ \downarrow & & \downarrow \lambda \\ \Lambda BF^* & \longrightarrow & THH(F) \end{array} \quad .$$

This is even a diagram of (parametrized) Γ-spaces.

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