The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$

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where the + denotes Quillen's plus construction. By definition, the total space and the base of the last fibration are components of \( K( R[G(S^m)] \) respectively \( K( R ) \).

The stable \( K \)-theory is so defined, that \( F' \) is an approximation to an \( m \)-fold delooping of \( K^S( R ) \). In particular, this makes \( K^S \) into a spectrum in a canonical way. We compute the stable homotopy of \( K( R[G(S^m)] \) relative to \( K(R) \) in two different ways. First note that the fibrations have sections. Since the total space in the second fibration has a product structure, we have a homotopy equivalence

\[
F' \times K(R) \simeq K( R[G(S^m)] \)
\]

For the relative stable homotopy we obtain

\[
\pi_{i+m}^S( K( R[G(S^m)] ), K(R) ) \simeq \pi_{i+m}^S( F' \wedge K(R) )
\]

Since \( F \) is \( m \)-connected, this equals the generalized homology of the space \( K(R) \) with coefficients in \( K^S(R) \), for small \( i \). In the limit over \( m \), we obtain equality.

But the spectrum \( K^S(R) \) is a module spectrum over \( R \), so it is a product of Eilenberg-MacLane spectra. The homology with coefficient in this spectrum is a sum of ordinary homology groups, with coefficients in the homotopy groups of \( K^S(R) \).

We can compute the relative stable homotopy in a different way, noticing that since stable homotopy is a homology theory, it does not change under the plus construction. This means, that we can use the first fibration to compute it. We obtain a spectral sequence converging to the relative stable homotopy. In the limit over \( m \), this spectral sequence collapses, and we obtain a formula

\[
\pi_{i+m}^S( B\hat{G}(R[G(S^m)]), B\hat{G}(R) ) \simeq H_i( GL(R), M(R) )
\]

For details, see [5], [11].

Combining our two calculations, we get

\[
H_k( GL(R), M(R) ) \simeq \bigoplus_{i+j=k} H_i( K(R), \pi_j( K^S(R) ) )
\]

In particular, assuming the conjecture that stable \( K \)-theory equals topological Hochschild homology, and recalling that by a computation of Quillen the higher homology of \( GL(\mathbb{Z}/p) \) with coefficients in \( \mathbb{Z}/p \) vanishes, we obtain

\[
H_i( GL(\mathbb{Z}/p), M(\mathbb{Z}/p) ) \simeq \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z}/p & i \text{ even} \end{cases}
\]
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§1. We are going to determine the Hochschild homology of the ringfunctors given by $X \mapsto \mathbb{Z}[X]$ respectively $X \mapsto \mathbb{Z}/p[X]$.

Recall from [2] that if $F(-)$ is a commutative ring functor, then we can define the topological Hochschild homology $THH(F)$. This is a hyper-$\Gamma$-space in the sense of [10] and [15]. This means that in particular, that it has a ringstructure up to homotopy. We can also make a ringspace out of $F$. Let $F$ denote the infinite loopspace $\lim \Omega^n F(S^n)$. This can be made into a ring up to homotopy, and there is a map $F \to THH(F)$. In particular, the spectrum obtained from the infinite loopstructure associated to the additive structure in $THH(F)$ is a module spectrum over $F$.

It follows that if $F$ is given as $F(X)=\mathbb{R}[X]$ for a commutative ring, then $THH(F)$ is a product of Eilenberg-MacLane spectra. The argument is, that using the unit map

$$S^0 \to \mathbb{R}[S^0]$$

we can construct a retraction of spectra

$$THH(\mathbb{R}) \to \mathbb{R} \wedge THH(\mathbb{R}) \to THH(\mathbb{R})$$

Smash product of an Eilenberg-MacLane spectrum with any spectrum is a product of Eilenberg-MacLane spectra. It follows, that $THH(\mathbb{R})$ is a retract of a product of Eilenberg-MacLane spectra. But then it is a product of Eilenberg-MacLane spectra itself.

Let $K(M,n)$ denote the Eilenberg-MacLane spectrum of dimension $n$, which corresponds to the $R$-module $M$. Let $\prod^\wedge$ denote restricted product.

Theorem 1.1. a) $THH(\mathbb{Z}/p) = \prod_{i=0}^\infty K(\mathbb{Z}/p, 2i)$

b) $THH(\mathbb{Z}) = K(\mathbb{Z},0) \times \prod_{i=1}^\infty K(\mathbb{Z}/i, 2i-1)$

c) The map $THH(\mathbb{Z},\mathbb{Z}) \to THH(\mathbb{Z}/p, \mathbb{Z}/p)$ is the product of the canonical map $K(\mathbb{Z},0) \to K(\mathbb{Z}/p, 0)$ with the Bockstein maps

$$K(\mathbb{Z}/pi, 2pi-1) \to K(\mathbb{Z}/p, 2pi)$$

d) We can choose the isomorphism in part a, so that if

$$\Delta^{2i} \in H^{2i}(K(\mathbb{Z}/p, 2i), \mathbb{Z}/p) \subset H^{2i}(THH(\mathbb{Z}/p), \mathbb{Z}/p)$$

is the fundamental class, then the coproduct in cohomology, given by the product in $THH$, is computed by the following formula

$$\Delta^{2i} = \sum_{j=0}^i \Delta^{2j} \otimes \Delta^{2(i-j)}$$
e) We can chose the isomorphism in part b so that if
\[ t_{2i-1} \in H^{2i-1}(K(\mathbb{Z}/1, 2i-1), \mathbb{Z}/p) \subset H^{2i-1}(\text{THH}(\mathbb{Z}/p), \mathbb{Z}/p) \]
is the fundamental class, then the coproduct is given by
\[ \Delta t_{2p^i-1} = 1 \otimes t_{2p^i-1} + t_{2p^i-1} \otimes 1 + \beta \left( \sum_{j=1}^{i-1} t_{2p^j-1} \otimes t_{2p(i-j)-1} \right) \]

Here \( \beta \) denotes the Bockstein associated to \( p \).

Remark: Part d simply asserts that the multiplicative structure is maximally nontrivial. This could also be formulated with homotopy groups (the ring of homotopy groups is a graded polynomial ring). Part e can also be formulated in terms of homotopy groups, but for this one needs homotopy groups with finite coefficients.

The proof of theorem 1.1 will occupy the rest of this section. We are going to compute the spectrum homology of the spectrum of the topological Hochschild homology. This will be done by spectral sequences. In order to compute the differentials, and to solve certain extension problems in these spectral sequences, we will need precise information about the homology of the spectrum. These computations will be done in §3.

The first remark to be done, is that as \( \text{THH}(\mathbb{R}) \) is a product of Eilenberg-MacLane spectra, its homotopy type is determined by its homology. Actually, it is even determined by the homology with coefficients in \( \mathbb{Z}/p \) for each \( p \), together with complete knowledge of all higher Bockstein maps.

Each Eilenberg-MacLane spectrum \( K(\mathbb{Z}/p^n, m) \) contributes two summands in the spectrum homology of the topological Hochschild homology, both isomorphic to \( \mathcal{A}/(p) \), the dual of the Steenrod algebra at \( p \) modulo the Bockstein. One copy is shifted in dimension by \( m \) and one copy by \( m+1 \). The two classes are related by the higher Bockstein associated to \( p^n \).

Fix a prime \( p \). From now on, all homology groups are with coefficients in \( \mathbb{Z}/p \). The simplicial structure of topological Hochschild homology provides us with a spectral sequence converging to its spectrum homology. Let us first consider the case \( \text{THH}(\mathbb{Z}/p) \). The \( E^1 \)-term of this spectral sequence is given by
\[ E^1_{p,*} \cong \mathcal{A} \otimes p^* \]

The first differential is given by the boundary maps of the simplicial object. These induce the boundary maps defining (ordinary) Hochschild homology \( H(\mathcal{A}) \) of \( \mathcal{A} \) acting on itself.

It follows, that \( E^2 \) is isomorphic to Hochschild homology of \( \mathcal{A} \) acting on itself. Recall from [4] that for a commutative ring \( S \),
\[ H(\mathcal{A}) \cong \text{Tor}_* S \otimes S(\mathcal{A}, S) \]
Recall from [11], that
\[ \mathcal{A} = \mathbb{Z}/2[\xi_1, \xi_2, \ldots], \quad \text{deg} \xi_i = 2^i - 1 \quad (p = 2) \]
\[ \mathcal{A} = \mathbb{Z}/p[\xi_1, \xi_2, \ldots] \otimes \mathbb{Z}/p[\tau_0, \tau_1, \ldots] / \tau_1^2 = 0 \]
\[ \text{deg} \xi_i = 2p^i - 2, \quad \text{deg} \tau_i = 2p^i - 1 \quad (p > 2) \]
The K"unneth formula applied to the complex defining Tor says that if $M_1$ and $M_2$ are bimodules over the rings $R_1$ respectively $R_2$ then

$$\text{Tor}_{R_1} \otimes_{R_2} (M_1 \otimes_{R_1} M_2, M_1 \otimes_{R_1} M_2) \cong \text{Tor}_{R_1} (M_1, M_1) \otimes \text{Tor}_{R_2} (M_2, M_2)$$

Let $\mathcal{A}'$ be defined by the formula

$$\mathcal{A}' = \mathbb{Z}/2 [\xi_1 \otimes 1 - 1 \otimes \xi_1, \xi_2 \otimes 1 - 1 \otimes \xi_2, \ldots] \subset \mathcal{A} \otimes \mathcal{A} \quad \text{p = 2}$$

$$\mathcal{A}' = \mathbb{Z}/p [\xi_1 \otimes 1 - 1 \otimes \xi_1, \ldots] \otimes \mathbb{Z}/p [\tau_0 \otimes 1 - 1 \otimes \tau_0, \ldots] \quad \text{p > 2}.$$

Then the K"unneth formula may be applied in the present situation, in view of the two maps given by the inclusion $\mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}$ and the diagonal map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, respectively. We obtain that

$$H(\mathcal{A} \otimes \mathcal{A}) \cong \mathcal{A} \otimes \text{Tor}_{\mathcal{A}'}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Using the K"unneth formula again, we can further decompose the Tor-factor in this tensor product. We obtain

$$H(\mathcal{A}) \cong \mathcal{A} [\lambda_1, \lambda_2, \ldots] / \lambda_i^2 = 0 \quad \text{deg } \lambda_i = (1, 2^i - 1) \quad (p = 2)$$

$$H(\mathcal{A}) \cong \mathcal{A} [\lambda_1, \lambda_2, \ldots] / \lambda_i^2 = 0 \otimes \Gamma(\gamma_1) \otimes \Gamma(\gamma_2), \ldots$$

$$\text{deg } \lambda_i = (1, 2p^i - 2); \text{deg } \gamma_i = (1, 2p^i - 1) \quad (p > 2).$$

The class $\gamma_i^{(a)}$ is represented by $1 \otimes \tau_i \otimes \tau_i \otimes \ldots \otimes \tau_i$ (where the tensor product has $a+1$ factors), and $\lambda_i$ by $1 \otimes \xi_1$.

The gamma-algebra $\Gamma(a)$ is defined as the vector space over $\mathbb{Z}/p$ with basis given by the symbols $a^{(i)}$, and equipped with a product given by $a^{(i)} a^{(j)} = (1 + i) a^{(i+j)}$. An exercise in binomial coefficients shows that

$$\Gamma(a) = \mathbb{Z}/p [a^{(p^0)}, a^{(p^1)}, \ldots] / (a^{(p^1)})^p = 0.$$

The spectral sequence is slightly different in the cases $p = 2$ and $p$ odd. In case $p = 2$, the multiplicative generators are all in filtration 1, so for dimensional reasons, all differentials vanish on them. Since the product is compatible with the simplicial filtration, this implies that all differentials vanish.

That is, $E^{\infty} = E^2$ in the spectral sequence, as a ring. Passing from $E^{\infty}$ to the spectrum homology, we have an extension problem. This problem is resolved by the following lemma, which we are going to prove in §3.

Lemma 1.2. Let $\overline{\lambda}_i \in H_*(\text{THH}(\mathbb{Z}/2); \mathbb{Z}/2)$ represent the permanent cycle $\lambda_i$. Then

$$\left( \overline{\lambda}_i \right)^2 = \overline{\lambda}_{i+1}$$

up to a nonzero factor, and counted modulo decomposables.

The fact that there are no differentials in the spectral sequence, proves
that the spectrum homology of $\text{THH}(\mathbb{Z}/2)$ is a free module over $\mathcal{A}$ with exactly one generator in each even degree. It follows that in the product of Eilenberg-MacLane spectra, homotopy equivalent to $\text{THH}(\mathbb{Z}/2)$, there is exactly one copy of each of the spectra $K(\mathbb{Z}/2, 2i)$, $i \geq 0$. That is, 1.1.a follows for $p = 2$.

1.1.d follows for $p = 2$ from lemma 1.2. By changing the homotopy equivalence of $\text{THH}(\mathbb{Z}/2)$ to the product of Eilenberg-MacLane spectra, we can arrange that $\lambda_i^2 = \lambda_{i+1}$, not only modulo decomposables or up to a constant. 1.1.d follows now from dualization. In case $p$ is odd, there are nontrivial differentials.

Lemma 1.3. For $1 < k < p-1$ the differential $d_k$ is identically zero, and

$$d_{p-1}(\gamma_i^{(p)}) = \lambda_{i+1} (\gamma_i^{(p-2)(p-2)} \ldots \gamma_i^{(p)})^{p-1}.$$ 

This will be proved in §3.

The ring $E^2$ is in this case generated by the classes $\lambda_i$ and $\gamma_i^{(p)}$. Since the classes $\lambda_i$ have filtration 1, all differentials $d_i$ for $i > 1$ vanish on them. The first $p-1$ differentials are therefore determined by lemma 1.3.

We want to compute $E^p$.

We can write the $E^{p-1}$ term as a tensor product:

$$E^{p-1} = A_p \otimes A_{p-1} \otimes \ldots$$

where $A_p = \mathcal{A}[\gamma_i] / \lambda_{i+1} \otimes \Gamma(\gamma_i)$.

The differential $d_{p-1}$ maps $A_p$ to itself, so we can consider the homology of $A_p$ with respect to it.

$A_p$ is the direct sum of two copies of $\mathcal{A} \otimes \Gamma(\gamma_i)$, indexed by 1 and $\lambda_{p-1}$. The differential maps one of the copies to the other. In each dimension congruent to 0 modulo 2p the ring $\mathcal{A} \otimes \Gamma(\gamma_i)$ has one copy of the vectorspace $\mathbb{Z}/p$. The differential decreases degree by 1. We claim that the differential is injective. This also proves, by dimension counting, that the kernel consists exactly of the elements of filtration less than $p$. To check the injectivity, note that it suffices to prove the nonvanishing of the differential on monomials in the symbols $\gamma_i^{(p)}$.

This follows directly from the formula for the differential.

The homology of $A_p$ with respect to $d_{p-1}$ equals

$$B_p = \mathcal{A}[\gamma_i] / (\gamma_i)^p.$$ 

The Künneth formula shows that

$$E^p = B_p \otimes B_{p-1} \otimes \ldots.$$ 

This ring has a set of generators in filtration less than or equal to 1. It follows that all higher differentials are zero. As in the case $p = 2$, this statement
Lemma 1.4. Let \( \gamma_1 \in H_* ( THH( \mathbb{Z}/p ) ; \mathbb{Z}/p ) \) represent the permanent cycle \( \gamma_1 \). Then, up to a factor, and modulo decomposables

\[
( \gamma_1 )^p = \gamma_{1+1}
\]

The proof will be given in §3.

In the same way as for \( p = 2 \), this proves 1.1. for odd \( p \).

We now turn to the spectrum \( THH(\mathbb{Z}) \). We fix a prime \( p \). The argument will be different in the two cases \( p = 2 \) and \( p \) odd.

As before, we have a spectral sequence with

\[
E^2 = H( \overline{A} )
\]

where \( \overline{A} = H_* ( \mathbb{Z} ; \mathbb{Z}/p ) \) is the spectrum homology of the Eilenberg-Maclane spectrum of the ring \( \mathbb{Z} \).

This is a spectral sequence of algebras over \( \overline{A} \), which are free as \( \overline{A} \)-modules.

We first treat the case \( p = 2 \).

The ring structure of \( \overline{A} \) is known, see [9]. It is a polynomial algebra over \( \mathbb{Z}/2 \) on one generator \( \eta \) of degree 2, and generators \( \xi_i \) of degree \( 2^i - 1 \) for each \( i \geq 2 \).

The ring map \( \mathbb{Z} \rightarrow \mathbb{Z}/2 \) induces a map \( \overline{A} \rightarrow A \). This map is given by

\[
\eta \mapsto \xi^2
\]
\[
\xi_i \mapsto \xi_i.
\]

There is a spectral sequence converging to the spectrum homology with coefficients in \( \mathbb{Z}/2 \) of \( THH(\mathbb{Z}) \). Using the reformulation of Hochschild homology as a Tor, and the Kunneth formula, we can compute that

\[
E^2 = \overline{A} \langle e_3, e_4, e_8, e_{16}, \ldots \rangle / (e_i)^2 = 0
\]

The class \( e_3 \) is given by \( 1 \otimes \eta \), and the class \( e_{2^i} \) by \( 1 \otimes \xi_i \).

The classes \( e_i \) all have filtration 1, so all differentials vanish, and \( E^\infty \) equals \( E^2 \).

We consider the multiplicative extensions in the \( E^\infty \). We claim, that we can choose representatives \( \overline{e}_{2^i} \) in \( H_* ( THH(\mathbb{Z}) ) \) of the classes \( e_{2^i} \), so that they are related by the extension

\[
( \overline{e}_{2^i} )^2 = \overline{e}_{2^{i+1}}.
\]

Under the map of spectral sequences induced by the simplicial map
the class $e_{2i}$ represented by $1 \otimes \bar{\epsilon}_i$ this maps to $1 \otimes \bar{\epsilon}_i$.

The map of $E^2$ terms

$$\mathcal{A} \{ e_3, e_4, e_8, \ldots \} / (e_2)^2 \to \mathcal{A} \{ \lambda_2, \lambda_4, \ldots \} / (\lambda_i)^2$$

sends $e_3$ represented by $1 \otimes \eta$ to zero, and $e_{2i}$ to $\lambda_{2i}$.

We have already solved the extension problem in $\text{THH}(\mathbb{Z}/2)$. We know, that we can choose classes $\bar{\lambda}_i$ representing $\lambda_i$ so that $(\bar{\lambda}_i)^2 = \bar{\lambda}_{2i}$. In particular, $H_*(\text{THH}(\mathbb{Z}/2))$ is a polynomial algebra. The image of $H_*(\text{THH}(\mathbb{Z}))$ is a subalgebra, containing the image of $\bar{\mathcal{A}}$ and the image of $\bar{e}_4$. Since $\bar{e}_4$ maps to $\bar{\lambda}_4$ modulo decomposables, the image of $\bar{e}_4$ is algebraically independent of $\bar{\mathcal{A}}$, and so algebraically independent of the image of $\bar{\mathcal{A}}$. It follows, that the image of $H_*(\text{THH}(\mathbb{Z}))$ in $H_*(\text{THH}(\mathbb{Z}/2))$ is a polynomial algebra, isomorphic to $\bar{\mathcal{A}}(\bar{e}_4)$. On the other hand, the square of $e_3$ is either equal to $\eta e_4$ or zero for dimensional reasons. The first possibility would contradict that $\text{THH}(\mathbb{Z})$ is a product of Eilenberg-MacLane spectra. There would be a nontrivial $k$-invariant, since $\text{Sq}^i(e_3) = 0$ would have square $\bar{e}_4$.

It follows, that $H_*(\text{THH}(\mathbb{Z}))$ contains $\mathcal{A} \otimes \mathbb{Z}/2(e_3, e_4) / (e_3)^2$. Counting dimensions, we conclude that this is indeed all of the homology. In particular, we can choose $\bar{e}_{2i}$ so that

$$(\bar{e}_{2i})^2 = \bar{e}_{2i+1}$$

We noted above, that $\text{THH}(\mathbb{Z})$ is homotopy equivalent to a product of Eilenberg-MacLane spectra. More precisely,

$$\text{THH}(\mathbb{Z}) = \mathbb{Z} \times \prod_{i} K( G_i, i )$$

where $G_i$ are finite groups. If we only ask for a 2-primary equivalence, we can assume that the groups $G_i$ are 2-groups.

The homology of the Eilenberg-MacLane spectrum $K(\mathbb{Z}/2^i, i)$ is a free module on two generators over $\bar{\mathcal{A}}$. The $E^{\infty}$-term above is also a free module over $\bar{\mathcal{A}}$. It has one generator for every dimension congruent to 0 or 1 modulo 4. Counting dimensions, we see that this can only be accounted for by a product

$$\text{THH}(\mathbb{Z})_{(2)} = \mathbb{Z}_{(2)} \times \prod_{i} K( \mathbb{Z}/2^i, 4i-1 )$$

We have to determine the numbers $l_i \geq 2$.

We claim that if $i = 2^j$, with odd $j$, then $l_i$ is at most $2^{i+1}$. Actually, the homology of $K(\mathbb{Z}/2^i, 4i-1)$ occurs in $E^{\infty}$ as the free module over $\bar{\mathcal{A}}$ generated by classes in dimensions $4i-1$ and $4i$. The generators are classes of the form $ae_3e_4 \ldots e_{2i+1}$, respectively $ae_{2i+2}$, where $a$ is the unique product
of multiplicative generators $e_{2r}$ of degree $2^{i+2}$ ($j-1$). Let $s$ be the number of
generators occurring in this product. Then the two generators have filtration $s+i'+1$
respectively $s+1$.

This means that the fundamental homology class in
$H_{4i-1}(K(\mathbb{Z}/l_1, 4i-1), \mathbb{Z}/2)$ is in the image of the map

$$H_{4i-1}(F_{s+i'+1}) \to H_{4i-1}(\text{THH}(\mathbb{Z})).$$

Let $\beta_{2r}$ denote the higher Bockstein, defined inductively on the kernel of
$\beta_{2r-1}$. These are the differentials in a spectral sequence converging to the tensor
product of $\mathbb{Z}/2$ with the 2-local homology modulo torsion. If $x$ is a class
in $H_*(\mathbb{Z}/2)$, which is the reduction of a class in $H_*(\mathbb{Z}/2^{p+1})$, then
$\beta_{2r}$ vanishes on the class $x$.

The fundamental class is the image of a higher Bockstein.
The nontrivial class in $H_{4i}(K(\mathbb{Z}/l_1, 4i-1))$ maps through $\beta_1$ to it.
In particular, this shows that there is an element in $H_*(G_{s+i'+1})$ where this higher
Bockstein is defined and nontrivial. This result can be improved, by noticing that
the class of dimension $4i$ in the Eilenberg-MacLane spectrum has filtration $s+1$, so
it is not in the image of

$$H_{4i}(F_s) \to H_{4i}(\text{THH}(\mathbb{Z})).$$

Combining these two statements, we see that the higher Bockstein $\beta_{l_1}$ is defined
on a nontrivial element of $H_*(F_{s+i'+i}/F_s)$.

But the quotient $F_{s+i'+i}/F_s$ is the suspension of a disjoint union of
smashproducts of Eilenberg-MacLane spectra, so relative to the suspension
of the space of components, its homology with coefficients in the 2-primary
localization of $\mathbb{Z}$ is 2-torsion. By induction, the torsion in the homology with
$Z_2$-coefficients of $F_{s+i'+i}/F_s$ is at most $2^{i+1}$-torsion. But then, the
higher Bockstein operation $\beta_{l_1+i}$ is only defined for the trivial element.

It follows that $l_1$ divides $2^{i+1}$, which is our claim. The next claim, is that
have an equality $l_1 = 2^{i+1}$. This is equivalent to the statement 1.1 b for
2-primary torsion. Recall from [2] that we have a product of
infinite loop spaces

$$\begin{array}{c}
\mathbb{Z}/2 \\ \times \\
\text{THH}(\mathbb{Z}) \end{array} \to \text{THH}(\mathbb{Z})$$

Let $C_* = C_*(\text{THH}(\mathbb{Z}))$ be the complex defining spectrum homology. Let
$W_*$ be the standard free resolution of $\mathbb{Z}$ over the groupring $\mathbb{Z}(\mathbb{Z}/2)$. This chain
complex has one generator $e_i$ in each dimension, as a chain complex over the groupring.

The map $\mu$ induces a map of chain complexes (for more details on this,
see the discussion of Dyer-Lashof operations on spectrum homology in §2)

$$\begin{array}{c}
\mu_* : W_* \otimes C_* \otimes C_* \to C_* \\
\end{array}$$

This map is invariant under the action of $\mathbb{Z}/2$.

Let $\tilde{x} \in C_*$ be a chain, representing a homology class with coefficients in
That is, there is a chain \( \tilde{y} \in C_\ast \), so that \( \delta \tilde{x} = 2^r \tilde{y} \). We define the Pontryagin square (see (6))

\[
P : H_n(C_\ast; \mathbb{Z}/2^r) \to H_{2n}(C_\ast; \mathbb{Z}/2^{r+1})
\]

by the formula \( P(x) = \mu_\ast(e_0 \otimes x \otimes x - 2^r e_1 \otimes y \otimes x) \).

Then, if \( \text{red} \) denotes reduction modulo \( 2^r \), \( \text{red}(P(x)) = x^2 \).

Using this explicit chain representing \( x^2 \), we obtain the following statement about homology with coefficients in \( \mathbb{Z}/2 \) and the higher Bockstein operations:

\[
\beta_{2r+1}(x^2) = \beta_2(x) \quad r > 1
\]

\[
\beta_2(x^2) = Q^n \beta_2(x)
\]

where \( Q^n(y) = \mu_\ast(e_1 \otimes y \otimes y) \).

We want to apply this to the classes \( \tilde{e}_i \). In §3, we prove

Lemma 1.5. \( Q^4(\tilde{e}_3) = 0 \).

Moreover, by the argument above, \( \ell_4 \leq 2 \), so it has to equal 2. It follows that \( \beta_2(\tilde{e}_4) = \tilde{e}_3 \). By the formulas above, using the Cartan formula for \( Q^6(e_3 e_4) \), and by our choice of \( e_{2i} \), we obtain inductively that

\[
\beta_{2j}(\tilde{e}_{2i}) = 0 \quad j < i - 2
\]

that is, \( \beta_{2j-1} \) is defined on \( \tilde{e}_{2i} \). Using that the higher Bocksteins are derivations, we obtain that \( \beta_{2j-1} \) is defined on a class representing the generator in dimension \( 2^j \). Our claim about \( \ell_4 \) follows, finishing the proof of 1.1 b. for 2-primary torsion.

The coproduct formula 1.1. e. follows from our computation of the multiplicative structure in \( H_\ast(\text{THH}(\mathbb{Z})) \). Choose the isomorphism with the product of Eilenberg-MacLane spectra so, that the generators \( \tilde{e}_i \) chosen above maps trivially into all the factors except one. Then \( \tau_{2i-1} \) is dual to \( e_3(e_4)^{i-1} \), and \( \beta(\tau_{2i-1}) \) is dual to \( (\tilde{e}_4)^i \). The formula follows on dualizing.

The case of odd torsion is similar, but involves differentials as an extra complication. In this case

\[
\tilde{A} = \mathbb{Z}/p[\tilde{\xi}_1, \tilde{\xi}_2, \ldots] \otimes \mathbb{Z}/p[\tilde{\tau}_1, \tilde{\tau}_2, \ldots] / (\tilde{\tau}_1)^2
\]

and the map \( \tilde{A} \to A \) is given by \( \tilde{\xi}_i \to \xi_i \), \( \tilde{\tau}_1 \to \tau_1 \).

The map of spectral sequences induced by the ring map \( \mathbb{Z} \to \mathbb{Z}/p \) is in this case on the \( E^2 \)-level an inclusion

\[
\tilde{A}(\lambda_1, \lambda_2, \ldots) / (\lambda_1)^2 \otimes \Gamma(\gamma_1) \otimes \ldots \subset A(\lambda_1, \lambda_2, \ldots) / (\lambda_1)^2 \otimes \Gamma(\gamma_0) \otimes \Gamma(\gamma_1) \otimes
\]

Since this map is injective, the first nontrivial differential in the spectral sequence of \( \text{THH}(\mathbb{Z}) \) is determined by the first nontrivial differential in the spectral
sequence of $\text{THH}(\mathbb{Z}/p)$. Recall from lemma 1.3:

$$d_{p-1}(\gamma_i^{(p^j)}) = \lambda_{i+1}(\gamma_i^{(p^{j-1})}) \gamma_i^{(p^{j-2})}).$$

In the same way as we did when we discussed the case $\text{THH}(\mathbb{Z}/2)$, we write

$$E_{p-1}(\text{THH}(\mathbb{Z}/p)) = A_1 \otimes A_2.$$  

$$A_1 = \overline{\mathbb{A}}[\lambda_1]/(\lambda_1)^2.$$  

$$A_i = \overline{\mathbb{A}}[\lambda_1]/(\lambda_1)^2 \otimes \Gamma(\gamma_{i-1}), \quad i \geq 2.$$  

Using the Künneth formula, and the computation of the homology of $A_i$ done above, we obtain that

$$E^p(\text{THH}(\mathbb{Z}/p)) \simeq B_1 \otimes B_2 \otimes \ldots.$$

where $B_1 = A_1$; $B_i \simeq \overline{\mathbb{A}}[\gamma_{i-1}]/(\gamma_{i-1})^P$.

All algebra generators of $E^p$ have filtration 1, so there can be no further differentials. We now have to solve the extension problems.

In this case, the target of the map

$$H_\ast(\text{THH}(\mathbb{Z})) \to H_\ast(\text{THH}(\mathbb{Z}/p))$$

is the tensor product of a polynomial algebra with an exterior algebra. In particular, since the image of $\gamma_1$ has a nontrivial square, not contained in $\overline{\mathbb{A}}$, it is algebraically independent of $\overline{\mathbb{A}}$. Moreover, a class representing $\lambda_1$ has a trivial square for dimensional reasons. By the same argument as in the case $p=2$, it follows that

$$H_\ast(\text{THH}(\mathbb{Z})) \simeq \overline{\mathbb{A}}[\overline{\lambda}_1, \overline{\gamma}_1]/(\overline{\lambda}_1)^2.$$  

Now the rest of the argument that we used in the case $p=2$ works. There is a $p$-primary equivalence

$$\text{THH}(\mathbb{Z}) \simeq \mathbb{Z} \times \prod_i K(\mathbb{Z}/l_i, 2p^i-1).$$

We have to determine the $p$-power $l_i$. By arguing with the filtration, we obtain that

$$p < l_i < p^{j+1},$$

where $i = j p^i$ (j prime to $p$). To conclude the proof of 1.1.b, we have to show that

$$l_i = p^{i+1}.$$  

Let $\mu$ be the product

$$\mu_* : \mathbb{E} \times \text{THH}(\mathbb{Z})^p \to \text{THH}(\mathbb{Z}).$$
Let $W_\ast$ be the standard free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/p]$, with one generator $e_i$ in each dimension $i$. $W_\ast$ is given by the formulas

$$
\delta e_{2i+1} = (1 - \lambda) e_{2i}
$$

$$
\delta e_{2i} = (1 + \lambda + \ldots + \lambda^{p-1}) e_{2i-1}
$$

The Pontryagin $p$th power

$$
P : H_n(THH(\mathbb{Z}); \mathbb{Z}/p^r) \to H_{np}(THH(\mathbb{Z}); \mathbb{Z}/p^{r+1})
$$

is given by

$$
P(x) = \mu_* \left( (e_0 \otimes x \ldots \otimes x) - p^r \left( \sum_{i=1}^{p-1} (e_i \otimes x \otimes x \ldots \otimes y \otimes x \ldots \otimes x) \right) \right).
$$

In the term indexed by $i$ in the sum, the factor $y$ occurs at the $i$th place.

If red denotes reduction modulo $p^r$, we have that $\text{red} \left( P(x) \right) = x^p$.

In this case, we obtain for homology with coefficients in $\mathbb{Z}/p$

$$
\beta_{p^i} \left( P(x) \right) = x^{p^i} \beta_{p^{i-1}} (x).
$$

As in the case $p=2$, this implies $l = p^{i+1}$.

\[ \text{§ 2. In this section, we start the proof of lemmas 1.3, 1.4 and 1.5. The method we use, is to examine the structure on the spectrum homology of THH(R) induced by the multiplicative structure on THH(R). We define Dyer-Lashof operations on this spectrum homology, which are related to the multiplicative structure. The evaluation of these, then gives information on the multiplicative structure of the space THH(R). In particular we can compute products of homology classes, and certain Dyer-Lashof operations. In order to extend the definition of these to the spectrum homology, we need to specify certain extra data, as will be explained below.}

In order to compute these operations, and also in order to prove the lemma on the differentials in the spectral sequence converging to the homology of THH(R), we compare the topological Hochschild homology to the simplicial spectrum $S^1_+ \wedge R$.

Recall from [2] that there is a map of simplicial spectra

$$
\lambda : S^1_+ \wedge R \to THH(R)
$$

Composing with the multiplication map of THH(R), we obtain a map of simplicial spectra

$$
\mathbb{E}Z/p_+ \wedge (S^1_+ \wedge R)^P \to THH(R)
$$

Of course, due to the usual problems with smash products of spectra, the last statement is not quite true. For our purposes, it is not necessary to pursue the question whether we can make it precisely true or not, because we can work with finite approximations.

Finally, we will also compute the differentials in the spectral sequence converging
to spectrum homology. This computation will also depend on comparison with a simpler spectral sequence. The ingredient needed to link the two spectral sequences is again the map $\lambda$ above.

Let $X$ be a space with a basepoint. We will only be concerned with "nice" spaces, for instance geometric realizations of simplicial sets.

We first describe the homology of the power construction.

Let $(x_i)$ be a basis of the homology of $X$ with coefficients in $\mathbb{Z}/p$. We fix a free action of $\mathbb{Z}/p$ on $S^{2r-1}$, so that the inclusion $S^{2r-1} \subset S^{2r+1}$ is equivariant. Then [8] the homology of the quotient of the $\mathbb{Z}/p$-action on $S^r \wedge X^p$ has a basis consisting of the classes

$$\left[ x_{i_1}, x_{i_2}, \ldots, x_{i_p} \right]$$

with the relations

$$\left[ x_{i_1}, x_{i_2}, \ldots, x_{i_p} \right] = \left[ x_{i_2}, x_{i_3}, \ldots, x_{i_p}, x_{i_1} \right]$$

$$\left( x_{i_1}, x_{i_2}, \ldots, x_{i_p} \right) = \left( x_{i_2}, x_{i_3}, \ldots, x_{i_p}, x_{i_1} \right).$$

The inclusion $S^{2r-1} \subset S^{2r+1}$ preserves the first two types of classes, and maps

$$\left( x_{i_1}, x_{i_2}, \ldots, x_{i_p} \right)$$

to zero, unless $i_1 = i_2 = \ldots = i_p$, in which case the class goes to

$$e_{2r-1} \otimes x_{i_1} \otimes \ldots \otimes x_{i_1}$$

We can form the direct limit of all spheres of odd dimension. As a limiting case we obtain, that

$$H_* \left( \mathbb{E}Z/p_+ \wedge X^{p} ; \mathbb{Z}/p \right)$$

has a basis consisting of classes

$$\left( 2.2 \right) \left[ x_{i_1}, x_{i_2}, \ldots, x_{i_p} \right] e_i \otimes \left( x_j \right) \otimes^p i \geq 0$$

with relations and degrees as in 2.1.

Now, assume that $X$ is an infinite loop space. Then there is a structure map

$$\left( 2.3 \right) \mu : \mathbb{E}Z/p_+ \wedge^r X^p \to X$$

If $a \in H_r \left( X ; \mathbb{Z}/p \right)$, one defines the Dyer-Lashof operation (eg in [7]) as the image of the class

$$e_{i-p-1} \otimes a \otimes^p \in H_{i+r} \left( \mathbb{E}Z/p_+ \wedge X^p ; \mathbb{Z}/p \right)$$
under the map of homology induced by $\mu$.

Now assume that $X$ has two product structures, which are commutative up to all higher homotopies. Assume that for each deloop $B^nX$ we have structure maps

$$\mu_n : \Sigma_\rho \wedge (B^nX)^\rho \to \tilde{B}^{np}X$$

which are $\Sigma_\rho$-equivariant with respect to the action permuting factors on the left side, and permuting desuspension coordinates cyclically right on the right side.

Also assume that these maps are related by commutative diagrams of $\Sigma_\rho$-equivariant maps

$$\begin{align*}
S^p \wedge \Sigma_\rho \wedge (B^nX)^\rho & \to \Sigma_\rho \wedge (\tilde{B}^{n+1}X)^\rho \\
\downarrow \text{id} \wedge \mu_n & \downarrow \text{id} \wedge \mu_{n+1} \\
S^p \wedge B^{np}X & \to \tilde{B}^{(n+1)p}X
\end{align*}$$

For instance, this is possible if $X$ arises as a hyper-$\Gamma$-space in the sense of [15].

We now introduce a further structure. The action of $\Sigma_\rho$ on $S^{np}$ given by permutation of coordinates, defines a spherical fibration over $B\Sigma_\rho$. This fibration is not trivial, but it is conceivable that it becomes fiberhomotopy trivial after restricting to a skeleton of $B\Sigma_\rho$.

According to [1], this indeed occurs. Given a natural number $m$, if a sufficiently high power of $p$ divides $m$, then the vector bundle defined by cyclic permutation of the coordinates in $\mathbb{R}^{mp}$ is trivial as a vector bundle on the $r$-skeleton $(B\Sigma_\rho)^r$. We choose a trivialization

$$t_n : (\Sigma_\rho \wedge S^{np}) \to S^{mp}$$

This trivialization can be used to trivialize certain other relevant fiber bundles. Let $\tilde{B}^{mp}X$ be the $(mp)$-fold deloop of $X$, considered as a $\Sigma_\rho$-space using the permutation of the coordinates in groups of $p$. Similarly, let $\Sigma_\rho$ act on the smash product $(B^{mp}X)^\rho$ by permutation of the factors of the smash product.

The suspension map $S^i \wedge B^{mp}X \to B^{mp+i}X$ induces equivariant maps

$$\begin{align*}
S^{np} \wedge (B^{mp}X) & \to (B^{mp+1}X)^\rho \\
S^{np} \wedge \tilde{B}^{mp}X & \to \tilde{B}^{(m+1)p}X
\end{align*}$$

These maps induce maps of fiber bundles over $\Sigma_\rho$.

Now, let $n$ be divisible by $m$.

We choose trivializations $t'_m$ of the bundles over $(B\Sigma_\rho)^r$ given by $\tilde{B}^{mp}X$, so that the trivializations are compatible with the pairings above. For instance, we can rewrite $\tilde{B}^{mp}X$ as

$$\Omega^{mp} \tilde{B}^{(m+N)p}X$$
with the appropriate action, and then trivialize this bundle, using \( t_m \).

These trivializations restrict to trivializations of \( \mathbb{Z}/p \)-bundles.

Combining these trivializations with the structure map, for each \( n \) divisible by \( m \) we obtain a map:

\[
\begin{align*}
f_n : ( E\mathbb{Z}/p_+ )_r \wedge ( B^n \mathcal{X} )^p & \to ( E\mathbb{Z}/p_+ )_r \wedge B^n \mathcal{X} \to B^n \mathcal{X}.
\end{align*}
\]

Using \( f_n \) we obtain a Dyer-Lashof operation

\[
\tilde{Q}^i : H_{n+r}( B^n \mathcal{X} ; \mathbb{Z}/p ) \to H_{n+p+r+i}( B^n \mathcal{X} ; \mathbb{Z}/p )
\]

by the formula

\[
(2.5) \quad \tilde{Q}^i(a) = f_n * ( e_{l-r(p-1)} \otimes a^p )
\]

Since the trivializations are chosen to be compatible with the stabilization \((2.4)\), the operations also commute with the homology suspension

\[
\sigma_m : H_*( B^r \mathcal{X} ) \to H_*( B^{r+m} \mathcal{X} )
\]

In particular, we can compare them to the usual Dyer-Lashof operations defined on \( \mathcal{X} \), without use of any trivializations. We obtain

\[
(2.6) \quad \tilde{Q}^i \sigma_m(a) = \sigma_m \tilde{Q}^i(a)
\]

Finally we obtain an operation defined on

\[
\lim H_*( B^r ; \mathbb{Z}/p )
\]

by choosing a mutually compatible family of trivializations \( t_{pr} \), one for each skeleton of \( B\mathbb{Z}/p \).

We now consider the map of simplicial spectra, defined in \([2]\)

\[
\lambda : S^1_* \wedge R \to \mathbb{THH}(R)
\]

We want to compute the map obtained from this using the multiplicative structure on \( \mathbb{THH}(R) \)

\[
\begin{align*}
\sum_{n^*} \Lambda^n ( S^1_* \wedge B^m R )^n & \to \sum_{n^*} \Lambda^n \mathbb{THH}(R)[m]^n \to \mathbb{THH}(R)[mn]
\end{align*}
\]

on the \( E^2 \)-level of the associated spectral sequences. The symbol \( \mathbb{THH}(R)[m] \) here means the \( m \)-fold delooping of the infinite loop space \( \mathbb{THH}(R) \), corresponding to the infinite loop structure obtained from the additive structure in \( \mathbb{THH}(R) \).

In order to do so, we first have to analyze the source of this map, that is, we have to analyze the simplicial set \( ( S^1 )^n \). We consider this set as the diagonal of the multisimplicial set obtained by taking product of \( n \) copies of the
standard simplicial $S^1$, the one with two nondegenerate simplices. The symmetric group $\Sigma_n$ acts on this simplicial set by permuting the simplicial directions.

Let $\tau_n$ be the simplicial space

$$ES_{n+} \Sigma^n_n \left( ( S^1 )^n_+ \wedge ( B^m_\mathbb{R} )^n \right)$$

The simplicial filtration on $T^n_+ = ( S^1 )^n_+$ lifts to a filtration

$$\tau_0 \subset \tau_1 \subset \ldots \subset \tau_n = \tau.$$  

This filtration induces a spectral sequence

$$E^2_{i,j}(\tau) \Rightarrow H_{i+j}(\tau).$$

The space $\tau_i / \tau_{i-1}$ can be described in terms of the orbits under $\Sigma_n$ of nondegenerate simplices in the torus $T^n$. It is a wedge of spaces to be described below, and the components of the wedge are indexed by such orbits.

The wedge component corresponding to a nondegenerate simplex $\sigma$ of $T^n$ is homeomorphic to

$$EH^r \wedge S^r \wedge ( B^m_\mathbb{R} )^n,$$

where $r$ is the dimension of the simplex, and $H$ is its isotropy group.

In particular, if $\sigma$ is in the unique orbit of nondegenerate $n$-cells, then the wedge component corresponding to $\sigma$ is equal to

$$S^n \wedge ( B^m_\mathbb{R} )^n.$$

We also consider the more general situation, where $\sigma$ is in the image of a torus, of dimension possibly less than $n$. Let

$$\varphi : [1,2,3,\ldots,n] \to [1,2,3,\ldots,i]$$

be a surjection of sets. Then $\varphi$ defines a diagonal map

$$\varphi_* : ( S^1 )^i \to ( S^1 )^n$$

by the formula $\varphi( x_1, x_2, \ldots ) = ( \varphi(x_1), \varphi(x_2), \ldots )$.

We can assume without loss of generality, that $\sigma$ is monotone increasing. The image of $T_i$ will not be invariant under the action of $\Sigma_n$. Let

$$H = \Sigma_{\varphi^{-1}(1)} \times \Sigma_{\varphi^{-1}(2)} \times \ldots \Sigma_{\varphi^{-1}(i)}.$$

Then $H$ is the isotropy group of $\sigma$. The normalizer $N(H)$ of $H$ in $\Sigma_n$ leaves the torus $T_i$ fixed as a set. It acts on the this torus through the map

$$W(H) = N(H)/H \to \Sigma_i.$$

The image of this map is the group of permutations, which leave invariant the function $\varphi^{-1}$ of cardinality $\varphi^{-1}(a)$, defined for $a \in [1,2,\ldots,i]$.

We can now describe the map $\lambda : \tau_i \to \text{THH}(R)\{mn\}$ on the quotients of the
filtration induced by the simplicial structure.

Lemma 2.7. The map is given in dimension $j$ on the wedge component corresponding to the orbit of $a$ in $(S^1)^n$ as the following composite:

\[
\begin{align*}
\Sigma_{\varphi^{-1}(1)} &\rightarrow S^1 \wedge B^m \rightarrow \cdots \rightarrow \Sigma_{\varphi^{-1}(2)} \rightarrow S^1 \wedge B^m
\end{align*}
\]

Proof. We first describe the multiplication map on $THH(R)$, following [2]. The simplicial infinite loopspace $THH(R)$ is given so that infinite loopspace in degree $r$ has a spectrum, which is a realization of the smash product of $r+1$ copies of the spectrum $R$. That is, we can approximate the spectrum by

\[
\Omega^{m(r+1)}( B^m \wedge (r+1) )
\]

The simplicial infinite loopspace $THH(R)^n$ is then in degree $r$ approximated by

\[
(\Omega^{m(r+1)}( B^m \wedge (r+1) ))^n
\]

The structure map $\mu$ is defined degreewise, and in degree $r$ it can be approximated by

\[
\Omega^{mn(r+1)}( B^m \wedge (r+1) ) \rightarrow \Omega^{mn(r+1)}( B^m \wedge (r+1) ) \rightarrow \Omega^{mn(r+1)}( B^m \wedge (r+1) )
\]

The spaces involved all carry a $\Sigma^n$-action. Using the trivializations we fixed, we can arrange that the maps extend to maps of bundles over skeletons of $B\Sigma^n$.

The map $S^1 \wedge B^m \rightarrow THH(R)[m]$ is degreewise given, up to homotopy, by the inclusion

\[
[r] \rightarrow ( B^m \vee B^m \vee \cdots ) \rightarrow \Omega^{mr}( B^m \wedge (r+1) )
\]

Here the component number $s$ in the wedge is included by the adjoint of the map $S^m \rightarrow B^m$ which includes $B^m$ as factor number $s$ in the smash product. The map from $T_1^n \wedge (B^m)^n$ to $THH(R)$ is the $n$th power of this map. To obtain the map of filtration quotients, we only have to compose these maps. First we have to identify the subspace of $\Sigma^n_1 / \Sigma^n_{j-1}$ corresponding to the simplex $a$. This will be a subspace of a smash product of wedges:

\[
( B^m \vee \cdots B^m ) \wedge (B^m)^n
\]

The subspace will be given, by picking one component of each of the factors. Choose component number one in the first $\varphi^{-1}(1)$ factors, component number two in the next $\varphi^{-1}(2)$ factors, and so on. A direct computation of the composite
restricted to this component then proves the lemma.

Corollary 2.8. Let \( a \in H_*(B^n \mathbb{R}) \). The image of \((\sigma_1 \otimes a \otimes \cdots \otimes a)\) represents

\[
(1 \otimes a \otimes a \otimes \cdots \otimes a)
\]

in \(E^2(THH(R))\). The tensor product contains \(n+1\) factors.

Proof. Apply 2.7. The group \( H \) is the trivial group, so the composite in 2.7 is just the inclusion of \( S^1 \bigwedge (B^m \mathbb{R})^{\wedge n} \) in \( F_1 THH(R)^{mn}/ F_{i-1} THH(R)^{mn} \).

Now we can compute the twisted Dyer-Lashof operations on the spectrum homology of \( THH(R) \). Let \( x \in \lim_m H_{i+m}(B^m \mathbb{R}) = \mathcal{R} \). Then the image of

\[
\sigma_1 \otimes x \in H_{i+m+1}(S^1 \bigwedge B^m \mathbb{R}) \text{ in the homology of } THH(R) \text{ represents the class } 1 \otimes x \in \mathcal{R} \otimes \mathcal{R} = \lim_{m} H_{i+m+1}(F_1 THH(R)^{m}/ F_0 THH(R)^{m}) \text{.}
\]

According to lemma 2.7, we have a commutative diagram

\[
\begin{array}{ccc}
E^\Sigma_n S^1 \bigwedge (B^m \mathbb{R})^{\wedge n} & \xrightarrow{\text{id} \wedge (\varphi_{1,n})^{\wedge n}} & E^\Sigma_n (THH(R)^{m})^{\wedge n} \\
\downarrow & & \downarrow \\
S^1 \bigwedge E^\Sigma_n (B^m \mathbb{R})^{\wedge n} & \xrightarrow{\Sigma} & \text{THH}(R)^{mn}
\end{array}
\]

In particular, we have

Lemma 2.9. \( \tilde{Q}^i(\lambda(\sigma_1 \otimes x)) = \lambda(\sigma_1 \otimes \tilde{Q}^i(x)) \)

The second problem which we have to solve, concerns the differentials in the spectral sequence. We want to compute

\[
d_{p-1}(1 \otimes x \otimes \cdots \otimes x)
\]

for certain \( x \in \mathcal{R} \). The interesting case is when the tensor product contains \(p^{i+1} \) factors.

The argument will be slightly different according to whether \( i = 1 \) or \( i > 1 \). In both cases, the idea of the proof is to compare \( THH(R) \) to the space

\[
E^\Sigma p^i S^1 \bigwedge (B^m \mathbb{R})^{\wedge p^i}.
\]

As a preliminary, we consider the spectral sequence associated to this space. Actually, we make an additional simplification. Consider a space \( X \) which is a suspension. We can form the simplicial space

\[
E_Z/\Sigma (S^1 \bigwedge (B^m \mathbb{R})^{\wedge p}).
\]

Since \( S^1 \bigwedge X \rightarrow S^1 \bigwedge X \) is a split surjection up to homotopy, the \( \Sigma_p \)-equivariant map
is a surjection, up to $\Sigma_p^{-}\text{equi}v$ariant homotopy. In particular, the class

$$e_0 \otimes (a_i \otimes x) \otimes \mathcal{H}_* (\mathbb{E}/p \wedge \mathbb{Z}/p \wedge (S_+ \wedge X)^{\wedge p})$$

is the image of the class

$$e_0 \otimes (a \otimes x) \otimes \mathcal{H}_* (\mathbb{E}/p \wedge \mathbb{Z}/p \wedge (S_+ \wedge X)^{\wedge p})$$

under a homotopy section. We are led to consider the space

$$\mathbb{E}/p_+ \wedge \mathbb{Z}/p \wedge (S_+ \wedge X)^{\wedge p}$$

with the filtration induced by the simplicial structure of $S^P = S^1 \wedge \ldots \wedge S^1$.

Inside this filtration, we have a shorter filtration, consisting of the three spaces

$$* \subset \mathbb{E}/p_+ \wedge \mathbb{Z}/p \wedge (S^1 \wedge X)^{\wedge p} \subset \mathbb{E}/p_+ \wedge \mathbb{Z}/p \wedge (S^P \wedge X)^{\wedge p}$$

Claim 2.12. The quotient of the two nontrivial spaces is homotopy equivalent to

$$S^2 \wedge (S_+^{P-2} \wedge X^P)$$

The boundary map is induced by the equivariant inclusion

$$S^{P-2} \subset S^\infty = \mathbb{E}/p$$

In particular, the boundary map on homology is determined by the remark after 2.1.

To see this, first check that the cofibre of the map

$$S^{P-2} \to S^0$$

which collapses the entire $(p-2)$-sphere to the basepoint, has cofibre equal to $S^{P-1}$. Suspending this cofibration, and forming the smash product with $\mathbb{E}/p$ we obtain a new cofibration

$$\mathbb{E}/p_+ \wedge S^1 \wedge X^{\wedge p} \to \mathbb{E}/p_+ \wedge S_+^{P-2} \wedge X^{\wedge p} \to \mathbb{E}/p_+ \wedge (S^2 \times \mathbb{Z}/p \wedge S_+^{P-2}) \wedge X^{\wedge p}$$

The claim follows from this and the observation, that as $\mathbb{Z}/p$ acts freely on $S^{P-2}$, the following projection is a $\mathbb{Z}/p$-equivariant homotopy equivalence.

$$S^{P-2} \times \mathbb{E}/p \to \mathbb{E}/p$$

We can compute the long exact sequence in homology, belonging to the filtration determined by 2.12. In the notation of 2.1, the differential

$$\delta : H_*(S^2 \wedge S_+^{P-2} \wedge X^{\wedge p} ; \mathbb{Z}/p) \to H_{*-1}(S^1 \wedge \mathbb{E}/p_+ \wedge X^{\wedge p} ; \mathbb{Z}/p)$$
is given by the formula
\[ \delta ( \sigma_2 \otimes e \otimes x^{\otimes p} ) = \sigma_1 \otimes e \otimes x^{\otimes p} . \]

In particular, the boundary of the class \( \sigma_2 \otimes e \otimes x^{\otimes p} \) is \( \sigma_1 \otimes e \otimes x^{\otimes p} \).

Applying the homotopy section, and noting that the simplicial filtration is a refinement of the short filtration, this shows that in the spectral sequence belonging to the simplicial structure of
\[ \mathbb{E}_2 \mathbb{Z}/p \wedge (S^1)^p \wedge X^p \]
there is a nontrivial differential, which maps the class in \( \mathbb{E}_2 \) projecting to \( \sigma_2 \otimes e \otimes x^p \) into a class projecting to \( \sigma_1 \otimes e \otimes x^p \).

The two classes will be given by the two classes
\[ \sigma_1 \otimes x^{\otimes p} \epsilon \mathbb{H}_* ( S^p \wedge X^p ) \]
\[ \sigma_1 \otimes e \otimes x^{\otimes p} \epsilon \mathbb{H}_* ( S^1 \wedge \mathbb{E}_2 \mathbb{Z}/p \wedge X^p ) \]

Now, let \( X \) be an approximation of \( B^mR \). The map
\[ \mathbb{E}_2 \mathbb{Z}/p \wedge (S^1)^p \wedge X^p \rightarrow \mathbb{E}_2 \mathbb{Z}/p \wedge \text{THH}(R)^p \rightarrow \text{THH}(R) \]
preserves simplicial filtration, so the classes above will map to two classes in \( \mathbb{E}_2 \text{THH}(R) \) which are related through a differential \( d_0 \).

Applying lemma 2.7, we obtain

Lemma 2.13. Let \( x \epsilon \mathbb{H}_n ( R; \mathbb{Z}/p ) \). Then we have the relation
\[ d_{p-1} ( 1 \otimes x \otimes \ldots \otimes x ) = 1 \otimes \mathbb{Q}^{np+p-n-1}(x) . \]

Now consider the general case. The symmetric group \( \Sigma_i \) has a p-Sylow subgroup \( S_i(p) \subset \Sigma_i \). This Sylow subgroup is abstractly isomorphic to an iterated wreath product:
\[ S_i(p) \cong \mathbb{Z}/p \mid \mathbb{Z}/p \mid \ldots \mid \mathbb{Z}/p . \]

This group acts on \( S^p_i \) by permutation of coordinates. The union of all fixed point sets of all nontrivial subgroups of \( S_i(p) \) is a union \( F_i \) of \( (p^i-1) \)-dimensional spheres. The quotient can, using the case \( i = 1 \) treated above, be described as a smash product
\[ (S^2 \wedge S^p)^{p^i-1} \]
where the action of \( S_i(p) \) is induced from the action of \( \mathbb{Z}/p \) on \( S^2 \wedge S^p \).

The quotient
\[ \mathbb{E}S_i^p (p) \mid S^p_i \wedge X^p \mid \mathbb{E}S_i^p (p) \mid F^p_i \wedge X^p \]
is homeomorphic to
\[ ES_i(\mathbb{P}) \cdot \left( S^2 \wedge S^{p-2} \right)^{p^{i-1}} \]

where the action on \( (S^2 \wedge S^{p-2})^{p^{i-1}} \) is induced from the action of \( \mathbb{Z}/p \) on \( S^2 \wedge S^{p-2} \). The cofibration giving rise to this quotient, corresponds to the short filtration in the case \( i = 1 \).

Alternatively, we can describe this quotient as the iterated power construction
\[ \mathbb{Z}/p \cdot \left( \mathbb{Z}/p \cdot \left( \ldots \left( \mathbb{Z}/p \cdot \left( S^2 \wedge S^{p-2} \right)^{p^{i-1}} \right)^{p^{i-2}} \right)^{p^{i-3}} \right)^{p^{1}} \]

The next highest quotient in the filtration of
\[ ES_i(\mathbb{P}) + S^{p^i} \wedge X^{p^i} \]

induced from the fixed point sets in \( S^{p^i} \) is the space
\[ (S^i \wedge \mathbb{Z}/p \cdot X^{p^i}) \wedge \left( (S^2 \wedge (S^{p-2} \wedge X^{p^i}))^{p^{(p-1)}} \right) \ldots \]

We consider the somewhat more general situation, where we have a cofibration
\[ X \to Y \to Z. \]

Then the p-power construction on \( Y \) has a filtration through the spaces
\[ C_i = \{(e, y_1, y_2, \ldots, y_p) \in \mathbb{Z}/p \wedge Y^p ; \text{ at most } i \text{ of } y_1, y_2, \ldots \text{ not in } X\} \]

Lemma 2.14. In this situation, if \( z \in H^*(Z; \mathbb{Z}/p) \), \( x = \delta z \in H^*(X; \mathbb{Z}/p) \), then
\[ d_2(e_0 \otimes z^p) = x \otimes z^{p-1}. \]

Here we have made the identifications
\[ E^2_{p, -1} \cdot (C) = H^*(\mathbb{Z}/p \wedge Z^p) \]
\[ E^2_{p-1, -1} \cdot (C) = H^*(X \wedge Z^{p-1}) \]

Proof. Pick a chain \( z \) in \( C^*_*(Y) \) which represents \( z \) after projection to \( C^*_*(Z) \). Then, \( \delta (e_0(z)) = (\delta z) \otimes z^{p-1} \) represents the boundary of \( e_0 \otimes z^p \) in \( H(C_{p-1}) \). The claim follows, after reduction to homology.

We apply 2.14 to the filtration given by the inclusion
\[ ES_i(\mathbb{P}) S_i(\mathbb{P}) + F^i \wedge X^{p^i} \subset ES_i(\mathbb{P}) S_{p^i} \wedge X^{p^i} \]

Inductively in \( i \), each such cofibration arises from the previous one as the inclusion \( C_{p-1} \subset C_p \). By repeated application of 2.14, the boundary of
\[ e_0 \otimes (e_0 \ldots (\sigma_2 \otimes e_{p-2} \otimes x \otimes \ldots) \otimes p^{p-2}) \ldots ) \otimes p^{1} \]
is given by
\[ \left( (\sigma_2 \otimes e_0 \otimes x^{0}) \otimes (e_0 \otimes (\sigma_2 \otimes x^{0})^{0})^{0} \otimes \ldots \right). \]

By an application of 2.12 we finally obtain that the boundary equals
\[ (\sigma_1 \otimes e_0 \otimes x^{0}) \otimes (e_0 \otimes (\sigma_2 \otimes x^{0})^{0})^{0} \otimes \ldots. \]

The rest of the argument is similar to the case \( i = 1 \). We obtain

Lemma 2.15. Let \( x \in H_n(\mathbb{R}; \mathbb{Z}/p) \). Let
\[ \gamma_1(x) = 1 \otimes x \otimes x \ldots \otimes x = \epsilon \mathcal{R}^{p^i+1}. \]

Then the differential in the spectral sequence converging to spectrum homology of topological Hochschild homology is given by
\[ d_{p-1}(\gamma_j(x)) = (1 \otimes Q^{np+p-1}(x)) (\gamma_j(x))^{p-1} \ldots (\gamma_{i-1}(x))^{p-1}. \]

In particular, \( d_i(\gamma_j(x)) = 0 \) for \( i \leq p-1 \).

We now specialize 2.9 and 2.15 to the cases \( R = \mathbb{Z}/p \) and \( R = \mathbb{Z} \). These applications depend on the determination of the relevant Dyer-Lashof operations for these rings. We collect these computations in the next section.

§3. In this paragraph we prove the technical lemmas 1.2 to 1.5.

We use the computations in §2 specialized to the case \( \mathbb{Z}/p \). These computations relate differentials and extension problems in the spectral sequence converging to spectrum homology of \( \text{THH}(\mathbb{Z}/p) \) to questions about the map
\[ \mu: \mathbb{E} \mathbb{Z}/p \times K(\mathbb{Z}/p, n)^{\wedge} \mathbb{P} \to K(\mathbb{Z}/p, np) \]
classifying the cup product.

Recall from §1 the classes \( \tau_i \) and \( \xi_i \). Let \( n \) be large enough, so that these are defined in the homology of \( K(\mathbb{Z}/p, n) \).

Lemma 3.1. If \( n \) is large enough (in dependence of \( i \)), then
\[ \mu_*(e_{p-2} \otimes \tau_1 \otimes \ldots \otimes \tau_i) = (\text{unit}) \xi_{i+1} + (\text{decomposable}) \quad ; \quad p \text{ odd} \]
\[ \mu_*(e_{p-1} \otimes \tau_1 \otimes \ldots \otimes \tau_i) = (\text{unit}) \tau_{i+1} + (\text{decomposable}) \quad ; \quad p \text{ odd} \]
\[ \mu_*(e_{i} \otimes \xi_1 \otimes \xi_i) = \xi_{i+1} + (\text{decomposable}) \quad ; \quad p = 2 \]

Proof. For \( p \) odd, let \( Q_0 = \beta \), the Bockstein, \( p^i \) the Steenrod powers, \( R_i = p^i \), and inductively
Then $Q_i$ is a primitive cohomology operation, dual to $\tau_i$, and $R_i$ is a primitive cohomology operation, dual to $\xi_i$.

For $p = 2$, let $M_0$ be the Bockstein, and let inductively

$$M_i = \text{Sq}^{2i} M_{i-1} + M_{i-1} \text{Sq}^{2i}.$$

We have to prove that $Q_{i+1}$ and $M_{i+1}$ evaluate nontrivially on the classes $(e_{p-2} \otimes \tau_1 \otimes \ldots \otimes \tau_i)$, $(e_{p-1} \otimes \tau_1 \otimes \ldots \otimes \tau_i)$ and $(e_1 \otimes \xi_1 \otimes \xi_i)$ respectively.

The case $i = 0$ is covered by the calculation of $P_1 = R_1$, and $M_1$ evaluate nontrivially on the classes $(e \otimes \xi \otimes \ldots \otimes \xi)$ and $(e_1 \otimes \xi_1 \otimes \xi_1)$ respectively. $p-2$

To prove these assertions, consider the projection

$$\mathbb{Z}/p \rightarrow \mathbb{Z}/p \otimes (K(\mathbb{Z}/p, 1)_+) \rightarrow \mathbb{Z}/p \otimes (K(\mathbb{Z}/p, 1))^{\wedge}$$

determined by a choice of basepoint. This map is injective on cohomology, so it suffices to prove our assertions for the source of the map.

In this space, we have a cup product decomposition
Using the Cartan formula for the primitive operation $Q_{i-1}$, and the relation that $p^i(x) = 0$ for $\deg(x)$ smaller than $2p^i$, since $i > 0$, so that $P^i(1) = 0$, we see:

\[ Q_i \left( e^0 \otimes \alpha \otimes \ldots \otimes \alpha \right) = P^i \left( Q_{i-1} - Q_{i-1} p^i \right) \left( e^0 \otimes \alpha \otimes \ldots \otimes \alpha \right) = e^{D^f} \otimes Q_{i-1} \otimes \ldots \otimes Q_{i-1} \cdot P^{k+2} \left( e^{D^f} \right) \]

This proves the first assertion in the list, since the case $i = 0$ is known. The other assertions follow in the same way.

To get from our assertions about classifying spaces $K(\mathbb{Z}/p, 1)$ to the lemma, we again use the multiplicative structure. There is a map

$$ f : (K(\mathbb{Z}/p, 1)^\wedge m) \wedge (K(\mathbb{Z}/p, 1)^\wedge n) \to K(\mathbb{Z}/p, m+2n) $$

classifying the cohomology class

$$ (\beta \otimes \beta \otimes \ldots \otimes \beta) \otimes (\beta \otimes \beta \otimes \ldots \otimes \beta) \]

This map is injective on cohomology in small dimensions. To see this, recall that the dual of the Steenrod algebra is generated by classes defined in $K(\mathbb{Z}/p, 1)$ and $K(\mathbb{Z}/p, 2)$. Thus, we only have to prove that

$$ Q_i \left( e^0 \otimes a \otimes a \otimes \ldots \otimes a \right) = \text{(unit)} e^{D^f} \otimes Q_{i-1} a \otimes Q_{i-1} a \otimes \ldots \otimes Q_{i-1} a \]

and similarily for $P_i$. This follows from our formulas for $n = 1$ and the Cartan formula.

In case $p = 2$, we note that the map

$$ K(\mathbb{Z}/2, 1)^\wedge n \to K(\mathbb{Z}/2, n) $$

is injective on homology in small dimensions, and use the Cartan formula.

We can now prove the lemmas in §1.

Proof of 1.2. and 1.4.

According to lemma 2.9 we have

$$ \left\{ \begin{array}{l}
\left[ \lambda \left( \sigma_i \otimes \xi_i \right) \right] \otimes^2 Q^{2i} = \lambda \left( \sigma_i \otimes Q^{2i} \left( \xi_i \right) \right) \\
\left[ \lambda \left( \sigma_i \otimes \tau_j \right) \right] \otimes^p = Q^{2p+1-2p} \left( \lambda \left( \sigma_i \otimes \tau_j \right) \right) = \lambda \left( \sigma_i \otimes Q^{2p+1-2p} \left( \tau_j \right) \right)
\end{array} \right. $$

The Lemma 3.1 says that $Q^{2i} \left( \xi_i \right) = \mu_{\ast} \left( e^0 \otimes \xi_i \otimes \xi_i \right) = \text{(unit)} \xi_i + \text{decomposables}$, and similarly for $\tau_j$. Since $\lambda \left( 1 \otimes \xi_i \right)$ is a particular choice of a class representing $\lambda_i$, Lemmas 1.2 follows. Similarly for 1.4.

Proof of 1.3. This follows from lemma 2.15, and the computation
\[ Q^{np+p-2n-2}(\tau_1) = \mu_{*}(e_{p-2} \otimes \tau_1 \otimes \tau_1 \otimes \ldots \otimes \tau_1) = \xi_1 \]

Proof of 1.5. This is again lemma 2.9, applied to the case

\[ Q^4(\varphi(\sigma_1 \otimes \eta)) = \varphi(\sigma_1 \otimes (Q^4(\eta))) \]

Since \(Q^4(\eta)\) has dimension 6, it is decomposable. It follows that \(\varphi(\sigma_1 \otimes (Q^4(\eta)))\) is trivial.

For later reference, we also note

Lemma 3.2. Let \(\lambda : S^1_+ \wedge \mathbb{Z} \to \text{THH}(\mathbb{Z})\) be the map of spectra discussed in §2. Then, the image of the homology class \(\sigma \otimes \xi_1\) unde \(\lambda\) represents

\[ 1 \otimes \xi_1 \in E^2(\text{THH}(\mathbb{Z})) \]

In particular, under the homotopy equivalence of theorem 1.1, the fundamental class in cohomology of \(K(\mathbb{Z}/p, 2p-1)\) pulls back to a class evaluating nontrivially on \(\sigma \otimes \xi_1\).

Proof. The first statement is a particular case of 2.7. The second statement follows from this and from the fact that the \(1 \otimes \xi_1\) generates \(E^2\) in this dimension (see the analysis of the spectral sequence in §1).
References