Stable Homotopy and Algebra over Spheres

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Abstract

In this talk, I will introduce spectra, the basic objects of stable homotopy theory, and explain how they are used to enlarge the category of rings to include objects of interest to algebraic topology.

In ordinary algebra, a ring is an algebra over the integers. In fact, the integers are initial among rings, insofar as any ring receives a unique map from the integers. In stable homotopy theory, the basic objects of study are stable homotopy groups, which are algebraic invariants of a space built from homotopy classes of maps out of spheres. Homotopy theorists then go one step further and define algebras over the sphere spectrum, by way of analogy to rings as algebras over the integers. These ring spectra, as they are called, are ubiquitous in modern homotopy theory.

1 INTRODUCTION

One of the biggest, if not *the* biggest, open problem in algebraic topology is computing the homotopy groups of spheres. So why do people care?

Say we're building a space Y by attaching a cell to X by an attaching map $f\colon S^{n-1} \to X$. This defines a pullback

$$\begin{array}{ccc} D^n & \dashrightarrow & Y \\ \uparrow & & \downarrow \\ S^{n-1} & \xrightarrow{f} & X, \end{array}$$

where the homotopy type of Y depends only on the homotopy type of f; all ways of attaching n-cells to X are determined by f and $\pi_{n-1}X$. If X is itself built from attaching cells to smaller cells, then the basic building blocks of spaces are sphere. So we can ask about their homotopy groups $\pi_n S^m$.

These are very hard to compute – it's still an open problem to determine them in general. For many reasons, the stable homotopy groups are easier to compute than the regular homotopy groups. This is the start of stable homotopy theory.

2 STABLE HOMOTOPY

Definition 2.1. Given two pointed topological spaces (X, x_0) and (Y, y_0), the **smash product** of X and Y is

$$X \wedge Y := {}^{X \times Y}\!\big/_{(X \times \{y_0\}) \cup (\{x_0\} \times Y)}.$$

Definition 2.2. The **(reduced) suspension** of a topological space X is the smash product of X with S^1:

$$\Sigma X := S^1 \wedge X$$

Example 2.3. The smash product of spheres is again a sphere:

$$S^n \wedge S^m = S^{n+m}.$$

The suspension of a sphere is a sphere of one larger dimension:

$$\Sigma S^n = S^{n+1}.$$

The 0-sphere $S^0 = \{0, 1\}$ is the unit for the smash product:

$$S^0 \wedge X \simeq X.$$

If we have a map $f\colon X \to Y$, we may suspend it to get a new map $\Sigma f\colon \Sigma X \to \Sigma Y$. In particular, this means that for $f\colon S^i \to X$, there is a new map $\Sigma f\colon \Sigma S^i = S^{i+1} \to \Sigma X$. Hence, we have a map

$$\pi_i(X) \to \pi_{i+1}(\Sigma X).$$

This map is fundamental to stable homotopy theory. The first important theorem in stable homotopy theory gives conditions for this to be an isomorphism.

Theorem 2.4 (Freudenthal Suspension Theorem)**.** *If X is n-connected, then $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2n$.*

Corollary 2.5. *If X is k-connected, then ΣX is $(k+1)$-connected.*

Corollary 2.6. *The sequence of groups*

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \pi_{i+3}(\Sigma^3 X) \to \cdots, \tag{1}$$

is eventually constant.

Proof. Assume that X is k-connected, where k may be as small as $k = -1$. The connectivity of the spaces $\Sigma^j X$ grows as $k + j$, while the index grows as $i + j$. For j sufficiently large, $2(k + j) > i + j$ and we may apply Freudenthal Suspension to conclude that

$$\pi_{i+j}(\Sigma^j X) \cong \pi_{i+j+1}(\Sigma^{j+1} X) \cong \pi_{i+j+2}(\Sigma^{j+2} X) \cong \cdots \qquad \square$$

Definition 2.7. The eventual constant in the sequence **??** is the **i-th stable homotopy group of** X,

$$\pi_i^S(X) := \operatorname*{colim}_j \pi_{i+j}(\Sigma^j X).$$

So in stable homotopy theory, we think not about the space X, but instead about the sequence of spaces $X, \Sigma X, \Sigma^2 X, \dots$ Why not make this more general?

Definition 2.8. A **spectrum** E is a sequence of pointed topological spaces $E_0, E_1, E_2, \dots$ with maps $\Sigma E_i \xrightarrow{\sigma_i} E_{i+1}$. A **morphism of spectra** $f\colon E \to E'$ is a sequence of maps $f_i\colon E_i \to E_i'$ such that the following diagram commutes for all i.

$$\begin{array}{ccc} \Sigma E_i & \longrightarrow & E_{i+1} \\ \big\downarrow{\scriptstyle \Sigma f_i} & & \big\downarrow{\scriptstyle f_{i+1}} \\ \Sigma E_i' & \longrightarrow & E_{i+1}' \end{array}$$

Definition 2.9. The **i-th (stable) homotopy group** of X is the direct limit of the system of groups

$$\pi_i(X_0) \to \pi_{i+1}(\Sigma X_0) \xrightarrow{(\sigma_1)_*} \pi_{i+1}(X_1) \to \pi_{i+2}(\Sigma X_1) \xrightarrow{(\sigma_2)_*} \pi_{i+2}(X_2) \to \cdots$$

Example 2.10. (a) The **sphere spectrum** $\mathbb{S}$ is the spectrum with n-th space S^n and maps the identity maps $\Sigma S^n \xrightarrow{\text{id}} S^{n+1}$.

(b) The **suspension spectrum** $\Sigma^\infty X$ of a space X is the sequence of iterated suspensions of X. By taking the zeroth space of any spectrum E, we get a functor from spectra to spaces that is left-inverse to the suspension spectrum functor, so Σ^∞ is faithful. Moreover, any map of spectra $\Sigma^\infty X \to \Sigma^\infty Y$ is determined by a map of spaces $X_0 \to Y_0$.

(c) The **Thom spectrum** MO has n-th space $\text{Th}(\gamma^n)$, where $\gamma^n \to \text{Gr}_n(\mathbb{R}^\infty)$ is the canonical bundle of n-planes in $\mathbb{R}^\infty$ and $\text{Th}(\gamma^n)$ is the **Thom space**, the quotient of the disk bundle by the sphere bundle. The stable homotopy groups of this spectrum classify cobordism classes of manifolds.

(d) The spectrum KU is the alternating sequence of spaces $\mathbb{Z} \times BU$, U, $\mathbb{Z} \times BU$, U, ..., where U is the infinite unitary group and BU is its classifying space. It is a fact that $\pi_i U = \pi_{i+1}(\mathbb{Z} \times BU)$ and a theorem (Bott periodicity) that $\pi_{i+2} U \cong \pi_i U$.

(e) For each abelian group A, there is an **Eilenberg-MacLane spectrum** HA with n-th space

$$HA_n = K(A, n)$$

where $K(A, n)$ is the **Eilenberg-MacLane space** with

$$\pi_i K(A, n) = \begin{cases} A & (i = n), \\ 0 & (i \neq n). \end{cases}$$

Definition 2.11. An Ω-spectrum is a spectrum E for which $\pi_n(E_i) \cong \pi_{n+1}(E_{i+1})$ for all n and all i.

Example 2.12. The spectra HA and KU are Ω-spectra.

Definition 2.13. Any Ω-spectrum E defines a generalized cohomology theory $h^n(X)$ by

$$h^n(X) := [X, E_n].$$

We can also go the other way.

Theorem 2.14 (Brown representability)**.** *Suppose that the sequence of functors h^n from spaces to abelian groups is a generalized cohomology theory. Then there is a corresponding Ω-spectrum E such that $h^n(X) = [X, E_n]$.*

So studying spectra is in a sense slightly more general than studying generalized cohomology theories.

Example 2.15. Ordinary cohomology with coefficients in A corresponds to the Eilenberg-MacLane spectrum HA. Topological K-theory corresponds to the spectrum KU.

3 Algebra over Spheres

Frequently, a cohomology theory has a ring structure. Cohomology of spaces is a ring, as is topological K-theory. So when does a spectrum give rise to a ring structure on cohomology?

This happens when the spectrum E is a **ring spectrum**. This means that there are maps of spectra $E \wedge E \to E$ and $\mathbb{S} \to E$ that associative and unital *up to homotopy*.

Unfortunately, this is where things get a bit hairy. It *is* possible to define the smash product of spectra using the definition we have, but this only has good properties (unit laws, associativity) up to homotopy equivalence. This makes it really difficult to do algebra.

The fix is to replace the category **Spectra** by the category of Γ-spaces. For any nonnegative integer n, let n^+ be the finite pointed set $n^+ = \{0, 1, 2, \ldots, n\}$ where 0 is the basepoint element.

Definition 3.1. A **Γ-space** A is a functor $A\colon$ **FinSet**$_*$ $\to$ **Spaces**$_*$ from finite pointed sets to pointed topological spaces that sends 0^+ to a point. A **morphism of Γ-spaces** is a natural transformation.

Example 3.2. (a) The inclusion of **FinSet**$_*$ as discrete pointed spaces is a Γ-space.

(b) For any n, we may define a Γ-space Γ^n to be the set of functions from n^+ to m^+ in **FinSet**$_*$.

(c) For any space X, the functor that sends n^+ to $n^+ \wedge X = X \vee X \vee \cdots \vee X$.

(d) For any topological abelian group A, there is a functor HA with $HA(n^+) = A^n$.

Why is replacing spectra with Γ-spaces a reasonable thing to do? This next theorem says that for the purposes of (stable) homotopy theory, these two categories are the same.

Theorem 3.3 (Bousfield-Friedlander)**.** *There is a Quillen equivalence between the category of Γ-spaces and the category of connective Spectra.*

In light of this theorem, we often use Γ-spaces and spectra interchangeably.

Example 3.4. Continuing the previous example.

(a) The functor $\mathbf{FinSet}_* \hookrightarrow \mathbf{Spaces}_*$ corresponds to the sphere spectrum.

(b) The functor $\Gamma^n = \mathrm{Hom}_{\mathbf{FinSet}_*}(n^+, -)$ corresponds to $\mathbb{S}^n$.

(c) The functor $n^+ \mapsto n^+ \wedge X$ corresponds to the suspension spectrum $\Sigma^\infty X$.

(d) The functor $HA(n^+) = A^n$ corresponds to the Eilenberg-MacLane spectrum HA.

So to define a smash product of spectra, we may instead define a smash product of Γ spaces. First, recall the universal property of the tensor product of two abelian groups A and B:

Definition 3.5. The **tensor product** of A and B is an abelian group $A \otimes B$ such that for any bilinear map $f\colon A \times B \to C$, there is a unique map $\widehat{f}\colon A \otimes B \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & C \\ \downarrow & \nearrow_{\exists ! \widehat{f}} & \\ A \otimes B & & \end{array}$$

We define the smash product of Γ-spaces by analogy to the tensor product of abelian groups. Let $\wedge\colon \mathbf{FinSet}_* \times \mathbf{FinSet}_* \to \mathbf{FinSet}_*$ be the functor that smashes two finite pointed sets (K, k_0) and (L, ℓ_0) as discrete topological spaces:

$$K \wedge L := {}^{K \times L}\!\big/_{(K \times \{\ell_0\}) \cup (\{k_0\} \times L)}.$$

Definition 3.6. Define the **external smash product** of two Γ-spaces A and B to be the functor

$$\begin{aligned} \mathbf{FinSet}_* \times \mathbf{FinSet}_* &\xrightarrow{A \widetilde{\wedge} B} \mathbf{Spaces}_* \\ (n^+, m^+) &\longmapsto A(n^+) \wedge B(m^+) \end{aligned}$$

Definition 3.7. The **smash product** of two Γ-spaces X and Y is the unique functor $A \wedge B\colon \mathbf{FinSet}_* \to \mathbf{Spaces}_*$ that makes the following diagram commute

$$\begin{array}{ccc} \mathbf{FinSet}_* \times \mathbf{FinSet}_* & \xrightarrow{A \widetilde{\wedge} B} & \mathbf{Spaces}_* \\ {\scriptstyle\wedge}\big\downarrow & \nearrow_{A\wedge B} & \\ \mathbf{FinSet}_* & & \end{array}$$

This is, in particular, defined on objects by

$$(A \wedge B)(n^+) = \operatorname*{colim}_{k^+ \wedge \ell^+ \to n^+} A(k^+) \wedge B(\ell^+).$$

Example 3.8. $\Gamma^n \wedge \Gamma^m = \Gamma^{nm}$

Remark 3.9. The universal property of this smash product means that, in order to define maps $A \wedge B \to C$, it suffices to define continuous maps of spaces $A(n^+) \wedge B(m^+) \to C(n^+ \wedge m^+)$ for all n and m.

This definition of smash product is analogous to the tensor product of chain complexes in a way that we now make precise.

We may in fact extend the assignment $A \mapsto HA$ to a functor

$$H\colon \mathbf{Ch}(\mathbb{Z}) \to \mathbf{Spectra}\,.$$

Proposition 3.10. *The functor* H *has a left-inverse and left-adjoint functor* $L\colon \mathbf{Spectra} \to \mathbf{Ch}(\mathbb{Z})$*,*

$$\mathrm{Hom}_{\mathbf{Spectra}}(X, HA) \cong \mathrm{Hom}_{\mathbf{Ch}(\mathbb{Z})}(LX, A).$$

This proposition lets us translate back and forth between these topological gadgets and chain complexes of abelian groups.

Proposition 3.11. $L(X) \otimes L(Y) \cong L(X \wedge Y)$ *and* $L(\mathbb{S}) \cong \mathbb{Z}$*.*

This says that in Γ-spaces, $\mathbb{S}$ plays the role of the integers. In fact, $\mathbb{S}$ is the unit for the smash product of spectra, that is,

$$\mathbb{S} \wedge X \cong X \cong X \wedge \mathbb{S}.$$

The smash product is also associative.

Theorem 3.12. *The category of Γ-spaces is a symmetric monoidal category with this definition of the smash product and unit $\mathbb{S}$.*

The existence of this associative, unital smash product now allows us to define rings, algebras and modules in the category of Γ-spaces (or, equivalently, in the category of spectra). Notice that an ordinary ring is a $\mathbb{Z}$-algebra, so we define a ring spectrum to be an $\mathbb{S}$-algebra.

Definition 3.13. A **ring spectrum** R is a Γ-space R together with morphisms $m\colon R \wedge R \to R$ and $i\colon \mathbb{S} \to R$ such that the following diagrams commute.

$$\begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{m \wedge id} & R \wedge R \\ \Big\downarrow{\scriptstyle id \wedge m} & & \Big\downarrow{\scriptstyle m} \\ R \wedge R & \xrightarrow{\quad m \quad} & R \end{array} \qquad \begin{array}{ccccc} R \wedge \mathbb{S} & \xrightarrow{id \wedge i} & R \wedge R & \xleftarrow{i \wedge id} & \mathbb{S} \wedge R \\ & {\scriptstyle \cong} \searrow & \Big\downarrow{\scriptstyle m} & \swarrow {\scriptstyle \cong} & \\ & & R & & \end{array}$$

Example 3.14. If M is a topological monoid, define $\mathbb{S}[M]$ by $\mathbb{S}[M](n^+) = n^+ \wedge M$. This is the suspension spectrum of M, but with different notation. The unit $\mathbb{S} \to \mathbb{S}[M]$ and multiplication maps $\mathbb{S}[M] \wedge \mathbb{S}[M] \to \mathbb{S}[M]$ are induced by the unit and multiplication in M.

Example 3.15. If A is a ring, then HA is a Γ ring with unit map $\mathbb{S} \to HA$ defined by the unit of A and multiplication given by the composite

$$HA \wedge HA \to H(A \otimes A) \xrightarrow{H(m)} HA$$

where the first map is defined by

$$\begin{array}{ccc} HA(n^+) \wedge HA(m^+) & \longrightarrow & H(A \otimes A)(n^+ \wedge m^+) \\ \Big\| & & \Big\| \\ A \otimes \mathbb{Z}^n \wedge A \otimes \mathbb{Z}^m & \longrightarrow & (A \otimes A) \otimes Z^{nm} \\ \Cup & & \Cup \\ a \otimes (x_1, \ldots, x_n) \wedge b \otimes (y_1, \ldots, y_m) & \longmapsto & (a \otimes b) \otimes (x_i y_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}} \end{array}$$

Definition 3.16. An R**-module spectrum** M over a ring spectrum R is a Γ-space M together with a morphism $a\colon R \wedge M \to M$ such that the following diagrams commute.

$$
\begin{array}{ccc}
R \wedge R \wedge M & \xrightarrow{\mathrm{id} \wedge a} & R \wedge M \\
{\scriptstyle m \wedge \mathrm{id}} \downarrow & & \downarrow {\scriptstyle a} \\
R \wedge M & \xrightarrow{\quad a \quad} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{S} \wedge M & \xrightarrow{i \wedge \mathrm{id}} & R \wedge M \\
& \searrow^{\cong} & \downarrow {\scriptstyle m} \\
& & M
\end{array}
$$

We may define the category R-**Mod** of module spectra over a ring spectrum R. Notice that the category of $\mathbb{S}$-module spectra is just the category of Γ-spaces.

Example 3.17. If an abelian group M is an A-module for a ring A, then HM is an HA-module.

Given a ring spectrum R, we may also define the smash product $M \wedge_R N$ of two R-module spectra M and N over R, by way of analogy to the tensor product of modules over rings other than $\mathbb{Z}$. This behaves well in the following sense.

Proposition 3.18. $L(M) \otimes_{L(R)} L(N) \cong L(M \wedge_R N)$

Wait, but what about $H\mathbb{Z}$? If $\mathbb{S}$ replaces $\mathbb{Z}$ in the category of spectra, what are $H\mathbb{Z}$ modules? Well, the following theorem says that $H\mathbb{Z}$-modules capture ordinary algebra over $\mathbb{Z}$ inside the category of Spectra.

Theorem 3.19. *For an abelian group (or chain complex) A, the categories HA-**Mod** and A-**Mod** are Quillen equivalent.*

So really, doing algebra over $\mathbb{S}$ is an enlargement of ordinary algebra over $\mathbb{Z}$. There are many rich new examples that come from algebra over $\mathbb{S}$.