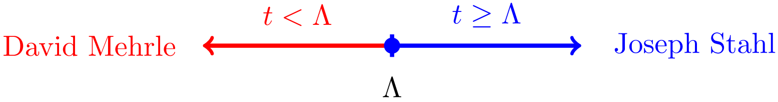
Newman's Conjecture for function field L-functions

Presenters:



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The Riemann Hypothesis and the ζ Function

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

$$\xi(s) = \xi(1-s), \quad \xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

Conjecture: Riemann Hypothesis

$$\xi(s) = 0 \implies \Re(s) = \frac{1}{2}$$

Pólya's idea

$$\Xi(x) = \xi\left(\frac{1}{2} + ix\right)$$

- If $x \in \mathbb{R}$, then $\Xi(x) \in \mathbb{R}$.
- Riemann Hypothesis true $\iff$ all zeros of $\Xi(x)$ are real.

$$\Xi \rightsquigarrow \Phi(u) = \frac{1}{2\pi}\int_0^\infty \Xi(x)\cos ux\,dx \rightsquigarrow \Xi_t(x) = \int_0^\infty e^{tu^2}\Phi(u)\cos ux\,du$$

Newman's Conjecture

Theorem (De Bruijn, Newman)

There exists $\Lambda \in \mathbb{R}$ such that if $t < \Lambda$, Ξ_t has a nonreal zero, and if $t \geq \Lambda$, Ξ_t has only real zeros.

$$\text{Riemann Hypothesis} \iff \Lambda \leq 0$$

Conjecture (Newman) $\Lambda \geq 0$.

"The new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so." - Newman

Fact

It is known that $\Lambda \geq -1.2 \cdot 10^{-11}$.

Function Field Analogy

Slogan: Function fields behave a lot like number fields!

<u>Number Fields and Function Fields</u>

Field	$K \quad (\mathbb{Q})$	$\mathbb{F}_q(T)$
Ring of Integers	$\mathcal{O}_K \quad (\mathbb{Z})$	$\mathbb{F}_q[T]$
Primes	$\mathfrak{p} \subseteq \mathcal{O}_K \quad ((p) \subseteq \mathbb{Z})$	$\pi \in \mathbb{F}_q[T]$ irreducible
Zeta Function	$\zeta_K \quad (\zeta_{\mathbb{Q}} = \zeta)$	Weil Zeta Function
	Riemann Hypothesis	Weil Conjectures

Newman setup in Function Fields

<u>Idea:</u> Mimic Pólya's setup

$$q = p^n, \quad D \in \mathbb{F}_q[x], \quad L(s,\chi_D) := \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s}$$

$$L(s,\chi_D) = \sum_{n=0}^{\infty} c_n \left(q^{-s}\right)^n, \qquad c_n = \sum_{\substack{f \text{ monic} \\ \deg f = n}} \chi_D(f)$$

$L(s,\chi_D)$ is a polynomial in q^{-s} of degree $\deg D - 1$.

$$L(s,\chi_D) \rightsquigarrow \xi(s,\chi_D) \rightsquigarrow \Xi(x,\chi_D) \rightsquigarrow \Xi_t(x,\chi_D)$$

Newman's Conjecture in Function Fields

Lemma (Andrade, Chang, Miller 2013)

If $\Xi_t(x, \chi_D)$ has only real zeros for some $t \in \mathbb{R}$, then for all $t' > t$, $\Xi_{t'}(x, \chi_D)$ has only real zeros.

Lemma (Andrade, Chang, Miller 2013)

There exists $\Lambda_D \in [-\infty, 0]$ such that

1. if $t \geq \Lambda$, then $\Xi_t(x, \chi_D)$ has only real zeros,
2. if $t < \Lambda$, then $\Xi_t(x, \chi_D)$ has a non-real zero.

Examples

Example

$D = x^5 + x^4 + x^3 + 2x + 2 \in \mathbb{F}_5[x]$:

$$\Xi_t(x, D) = 10e^{4t} \cos 2x - 2\sqrt{5}e^t \cos x - 1$$

$$\Lambda_D \approx -0.188565066$$

Example

$$D(T) = T^3 + T \in \mathbb{F}_3[T] \implies \Xi_t(x, \chi_D) = \sqrt{3}e^t \cos x$$

$$\Lambda_D = -\infty$$

Newman's Conjecture in Function Fields

Conjecture

Fix q a power of an odd prime. Then

$$\sup_{D \in \mathbb{F}_q[T] \text{ good}} \Lambda_D \geq 0.$$

Conjecture

Fix $g \in \mathbb{N}$. Then

$$\sup_{\substack{D \text{ good}, \ \deg D = 2g+1 \\ q = p^k, \ p \geq 3}} \Lambda_D \geq 0.$$

Newman's Conjecture in Function Fields

Conjecture

Fix $D \in \mathbb{Z}[T]$ square-free. Let p be prime, and let $D_p \in \mathbb{F}_p[T]$ be the polynomial obtained by reducing D modulo p. Then

$$\sup_{\substack{D_p \text{ good} \\ p \geq 3}} \Lambda_{D_p} \geq 0.$$

Previous Work

Theorem (Andrade, Chang, Miller 2013)

Let $D \in \mathbb{Z}[x]$ be square-free with $\deg D = 3$. For each odd prime p, we can reduce D to $D_p \in \mathbb{F}_p[x]$. Then $\sup_p \Lambda_{D_p} = 0$.

PROOF SKETCH.

Step 1: Show that

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$

where $a_p(D)$ is the trace of Frobenius.
Step 2: Use the Sato–Tate conjecture. $\square$

Motivation For Our Strategy

- If $q = p^n$ is a square, then $2\sqrt{q} \in \mathbb{Z}$, so $|a_q(D)|$ can actually equal $2\sqrt{q}$. In this case, $\Lambda_D = \log 1 = 0$.

- Weil conjectures $\implies \mathcal{E}/\mathbb{F}_p$ with the average number of points will acheive the maximum and minimum number of points possible over particular extensions of $\mathbb{F}_p$.

- Judicious choices of D and p (such that $y^2 = D(x)$ has $p+1$ points over $\mathbb{F}_p$) will give us Newman's conjecture in certain cases!

The Weil Conjectures for Curves

Theorem (Weil Conjectures)

Let X be a curve over $\mathbb{F}_q$. The Hasse-Weil zeta function of X is defined as $Z(X,s) = \exp\left(\sum_{m\geq 1} \frac{N_m}{m}\left(q^{-s}\right)^m\right)$, where N_m is the number of points of X over $\mathbb{F}_{q^m}$.

1. $$Z(X,s) = \frac{P(T)}{(1-T)(1-qT)}, \qquad P \in \mathbb{Z}[T]$$
2. $Z(X,n-s) = \pm q^{f(X)} Z(X,s)$.
3. Let α be a root of P. Then $|\alpha| = q^{-1/2}$.

Example:

$$Z(\mathbb{P}^1,s) = \frac{1}{(1-T)(1-qT)}$$

A Key Lemma and a Key Observation

Lemma (Andrade, Chang, Miller 2013)

$\Lambda_D = 0 \iff L(s, \chi_D)$ has a double root.

Observation

$L(s, \chi_D)$ is the numerator of the zeta function $Z(X, s)$,
$X : y^2 = D(x)$. More precisely, $Z(X, s) = Z(\mathbb{P}^1, s)L(s, \chi_D)$.

Proof (Idea)

Use the Euler products for $Z(X, s)$, $L(s, \chi_D)$.

$$Z(X, s) = \prod_{\pi \text{ monic, irred.}} (1 - N(\pi))^{-s}$$

$$L(s, \chi_D) = \prod_{\pi} \left(1 - \chi(\pi)N(\pi)^{-s}\right)^{-1}$$

Results

Theorem

The L-function corresponding to $D(x) = x^q - x$ has a double root. This implies that $\Lambda_D = 0$ (considering D over $\mathbb{F}_q$).

Proof Sketch

The curve $X : y^2 = x^q - x$ carries an action of $\mathbb{F}_q$ that commutes with Frobenius. These actions reduce to actions at the level of cohomology $H_\ell^*(X)$. For $X : y^2 = x^q - x$, $Z(X, s) = Z(\mathbb{P}^1, s)L(s, \chi_D)$. Next, recall that the L-function is defined as a Gauss sum. A result of Nick Katz $\implies L(s, \chi_D) = (T^2 q \pm 1)^g$.

Results

Corollary

If $\mathcal{F}$ is a family of good polynomials over various finite fields, and contains at least one polynomial over $\mathbb{F}_q$ of the form $x^q - x$ for some q, then

$$\sup_{D \in \mathcal{F}} \Lambda_D = 0.$$

In particular, $\mathcal{F} = \{D \in \mathbb{F}_q[T] \mid D \text{ good}\}$ and $\mathcal{F} = \{D \mid \deg D = 2g+1, 2g+1 = p^k \text{ for some } p\}$ are such families.

This sup is really a max!

Continuing Previous Results

Theorem

Let $D \in \mathbb{Z}[T]$ be a square-free monic cubic polynomial. Then there exists a number field $K/\mathbb{Q}$ such that

$$\sup_{\mathfrak{p} \subseteq \mathcal{O}_K} \Lambda_{D_\mathfrak{p}} = \max_{\mathfrak{p} \subseteq \mathcal{O}_K} \Lambda_{D_\mathfrak{p}} = 0,$$

where $D_\mathfrak{p}$ denotes reduction modulo the prime ideal $\mathfrak{p}$.

Future Directions

- Degree greater than 3 case of the third Newman's conjecture.

- Fix a number field $K/\mathbb{Q}$ and a square-free monic cubic $D \in \mathcal{O}_K[T]$. Does there exist a prime $\mathfrak{p} \subseteq \mathcal{O}_K$ such that $\Lambda_{D_\mathfrak{p}} = 0$ or a sequence of primes $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$ such that $\Lambda_{D_{\mathfrak{p}_i}} \to 0$?

- Does Newman's conjecture hold for the family $\mathcal{F} = \{D \mid \deg D = 2g+1, g \in \mathbb{N}\}$ when $2g+1$ is not a power of a prime?

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