REVIEW

MATH 1910 Recitation

§5.3 (Indefinite Integrals); §5.4, §5.5 (FTC)

September 6, 2016

(1) F is called an **antiderivative** of f if $\boxed{F'(x) = f(x)}$ (1) .

(2) Any two antiderivatives of f on an interval (a, b) differ by a constant.

(3) **Fundamental Theorem of Calculus, Part I (FTC I):** if $F(x)$ is an antiderivative for $f(x)$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$ (2)

(4) (a) $\int 0\,dx = \boxed{C}$ (3)

(b) $\int k\,dx = \boxed{kx + C}$ (4)

(c) $\int cf(x)\,dx = \boxed{c\int f(x)\,dx}$ (5)

(d) $\int (f(x) + g(x))\,dx = \boxed{\int f(x)\,dx}$ (6) $+ \boxed{\int g(x)\,dx}$ (7)

(e) $\int x^n\,dx = \boxed{\frac{x^{n+1}}{n+1} + C}$ (8)

(f) $\int \sin x\,dx = \boxed{-\cos x + C}$ (9)

(g) $\int \sec^2 x\,dx = \boxed{\tan x + C}$ (10)

(h) $\int \sec x \tan x\,dx = \boxed{\sec x + C}$ (11)

(5) To solve an initial value problem $^{dy}/_{dx} = f(x)$, $y(x_0) = y_0$, first find the general antiderivative $y = F(x) + C$. Then determine C using the initial condition $F(x_0) + C = y_0$.

(6) The **area function** with lower limit a is $A(x) = \boxed{\int_a^x f(t)\,dt}$ (12) .

(7) **Fundamental Theorem of Calculus, Part II (FTC II):**

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$ (13)

(8) A consequence of FTC II is that *every* continuous function has an antiderivative.

(9) Let $G(x) = \int_a^{g(x)} f(t)\,dt$. Let $A(x) = \int_a^x f(t)\,dt$. Then

$$\frac{d}{dx}G(x) = \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \boxed{\frac{d}{dx}A(g(x)) = A'(g(x))g'(x) = f(g(x))g'(x)}$$ (14)

SOLUTIONS

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(1) Evaluate the integral:

(a) $\int \cos x \, dx$

SOLUTION: $\int \cos x \, dx = \sin x + C$

(b) $\int \csc x \, \cot x \, dx$

SOLUTION: $\int \csc x \, \cot x \, dx = -\csc x + C$

(c) $\int \frac{3}{x^{3/2}} \, dx$

SOLUTION: Since $\frac{3}{x^{3/2}} \, dx = 3x^{-3/2}$, we get

$$\begin{aligned} \int \frac{3}{x^{3/2}} \, dx &= \int 3x^{-3/2} \, dx \\ &= 3\left(\frac{-1}{(-1/2)} x^{-1/2}\right) + C \\ &= -6x^{-1/2} + C \end{aligned}$$

(d) $\int_{-2}^{2} (10x^9 + 3x^5) \, dx$

SOLUTION: $\int_{-2}^{2} (10x^9 + 3x^5) \, dx = \left(x^{10} + \frac{1}{2}x^6\right)\Big|_{-2}^{2} = \left(2^{10} + \frac{1}{2}2^6\right) - \left(2^{10} + \frac{1}{2}2^6\right) = 0$

(e) $\int_{0}^{4} \sqrt{x} \, dx$

SOLUTION: $\int_{0}^{4} \sqrt{x} \, dx = \int_{0}^{4} x^{1/2} \, dx = \frac{2}{3}x^{3/2}\Big|_{0}^{4} = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$

(f) $\int_{\pi/4}^{3\pi/4} \sin\theta \, d\theta$

SOLUTION: $\int_{\pi/4}^{3\pi/4} \sin\theta \, d\theta = -\cos\theta\Big|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

(g) $\int_0^5 |x^2 - 4x + 3|\, dx$

SOLUTION: Write the integral as a sum of integrals without absolute values and then apply FTC I.

$$\begin{aligned}
\int_9^5 |x^2 - 4x + 3|\, dx &= \int_0^5 |(x-3)(x-1)|\, dx \\
&= \int_0^1 (x^2 - 4x + 3)\, dx + \int_1^3 (-x^2 - 4x + 3)\, dx + \int_3^5 (x^2 - 4x + 3)\, dx \\
&= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_1^2 + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_3^5 \\
&= \left(\frac{1}{3} - 2 + 3\right) - 0 - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{125}{3} - 50 + 15\right) - (9 - 18 + 9) \\
&= \frac{28}{3}
\end{aligned}$$

(h) $\int_4^9 \frac{16+t}{t^2}\, dt$

SOLUTION: $\int_4^9 \frac{16+t}{t^2}\, dt = \int_4^9 16t^{-2} + t^{-1}\, dt = -16t^{-1} + \log t\Big|_4^9 = \frac{20}{9} + \log\frac{9}{4}$

(2) Solve the differential equation $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$ with initial condition $y(1) = 1$.

SOLUTION: Since $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$, then

$$y = \int (8x^3 + 3x^2 - 3)\, dx = 2x^4 + x^3 - 3x + C$$

Thus $1 = y(1) = 0 + C$, and so $C = 1$. Therefore, $y = 2x^4 + x^3 - 3x + 1$.

(3) Given that $f''(x) = x^3 - 2x + 1$, $f'(0) = 1$, and $f(0) = 0$, find f' and then find f. SOLUTION: Let $g(x) = f'(x)$. The statement gives that $g'9x) = x^3 - 2x + 1$, $g(0) = 1$. From this initial value problem, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$. Then $g(0) = 1$ gives $C = 1$, so $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1$.

Now we have a new initial value problem to find f, namely $f'(x) = \frac{1}{4}x^4 - x^2 + x + 1$ and $f(0) = 0$. So we get that $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$. Then $f(0) = 0$ gives $C = 0$, so

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x.$$

(4) If $G(x) = \int_1^x \tan t\, dt$, find $G(1)$ and $G'(\pi/4)$.

SOLUTION: By definition, $G(1) = \int_1^1 \tan t\, dt = 0$. By FTC II, $G'(x) = \tan x$, so $G'(\pi/4) = \tan(\pi/4) = 1$.

(5) Find a formula for the function represented by the integral: $\int_2^x (t^2 - t)\, dt$.

SOLUTION: $\displaystyle\int_2^x (t^2 - t)\, dt = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right)\Big|_2^\pi = \frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{2}{3}$

(6) Express the antiderivative $F(x)$ of $f(x)$ as an integral, given that $f(x) = \sqrt{x^4+1}$ and $F(3) = 0$.

SOLUTION: The antiderivative $F(x)$ of $f(x) = \sqrt{x^4+1}$ satisfying $F(3) = 0$ is

$$F(x) = \int_3^x \sqrt{t^4+1}\, dt$$

(7) Calculate the derivative: $\displaystyle\frac{d}{dx}\int_1^{x^3} \tan t\, dt$.

SOLUTION: By combining FTC II and the chain rule. Let $G(x) = \displaystyle\int_1^{x^3} \tan t\, dt$, $A(x) = \displaystyle\int_1^x \tan t\, dt$, $g(x) = x^3$. Then $G(x) = A(g(x))$, so we can use the chain rule.

$$G'(x) = A'(g(x))g'(x) = \tan x^3(3x^2) = 3x^2 \tan x^3$$