One-Page Review

MATH 1910 Recitation

§6.1, §6.2, §6.3 (Areas, Volumes, Revolution)

September 20, 2016

(1) The graph of $x = f(y)$ is the graph of $y = f(x)$ reflected across the line $\boxed{y = x}$ [1].

(2) The area between $f(x)$ and $g(x)$ from a to b is $\boxed{\int_a^b (y_{\text{top}} - y_{\text{bottom}})\, dx}$ [2].

(3) The $\boxed{\text{average value}}$ [3] of $f(x)$ over the interval $[a, b]$ is $\boxed{\frac{1}{b-a}\int_a^b f(x)\, dx}$ [4].

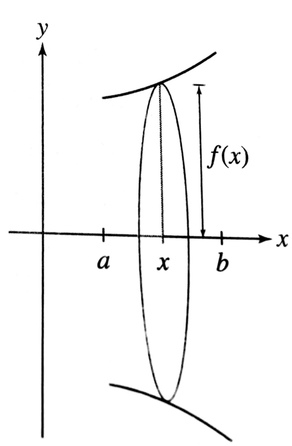
(4) The **Mean Value Theorem for Integrals** says that if f is continuous on $[a, b]$ with mean value M, then there is some $c \in [a, b]$ such that $\boxed{f(c) = M}$ [5].

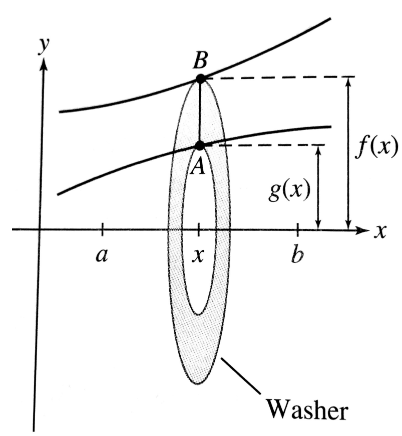
(5) If a shape has cross-sectional area $A(y)$ and height extends from $y = a$ to $y = b$, then it's volume is $\boxed{\int_a^b A(y)\, dy}$ [6].

(6) **Cavilieri's Principle** says if two solids have equal $\boxed{\text{cross-sectional areas}}$ [7], then they also have equal $\boxed{\text{volumes}}$ [8].

(7) **The Disk Method:** If $f(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region under the graph around the x-axis has volume $\boxed{\int_a^b \pi f(x)^2\, dx}$ [9].

(8) **The Washer Method:** If $f(x) \geq g(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region between $f(x)$ and $g(x)$ around the x-axis has volume $\boxed{\int_a^b \pi(f(x) - g(x))^2\, dx}$ [10].





Handout format by Drew Zemke

SOLUTIONS

MATH 1910 Recitation

§6.1, §6.2, §6.3 (Areas, Volumes, Revolution)

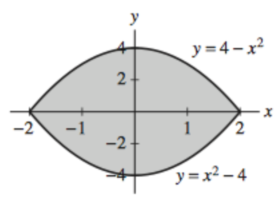
September 20, 2016

(1) Sketch the region enclosed by the curves and set up an integral to compute it's area, but do not evaluate.

(a) $y = 4 - x^2$, $y = x^2 - 4$

SOLUTION: Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4) \right) dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left(8x - \frac{2}{3}x^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



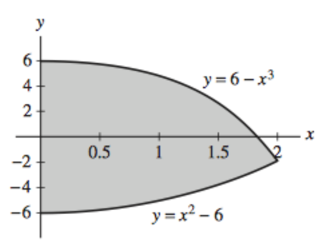
(b) $y = x^2 - 6$, $y = 6 - x^3$, $x = 0$

SOLUTION: Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval $[0, 2]$; hence, the area of the region enclosed by these curves and the y-axis is

$$\int_{0}^{2} \left((6 - x^3) - (x^2 - 6) \right) dx = \int_{0}^{2} (-x^3 - x^2 + 12)\, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x \right) \Big|_{0}^{2} = \frac{52}{3}$$



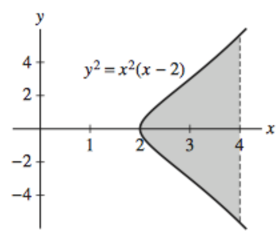
(c) $y = x\sqrt{x-2}$, $y = -x\sqrt{x-2}$, $x = 4$

SOLUTION: Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x-2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_2^4 \left(x\sqrt{x-2} - (-x\sqrt{x-2})\right)\,dx = \int_2^4 2x\sqrt{x-2}\,dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

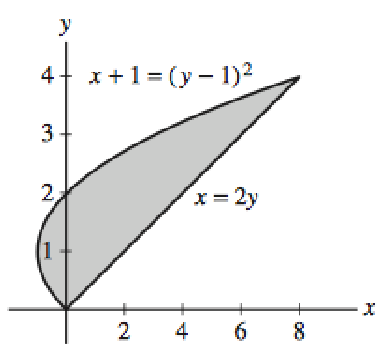
$$\int_2^4 2x\sqrt{x-2}\,dx = \int_0^2 2(u+2)\sqrt{u}\,du = \int_0^2 \left(2u^{3/2} + 4u^{1/2}\right)\,du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right)\Big|_0^2 = \frac{128\sqrt{2}}{15}.$$



(d) $x = 2y$, $x + 1 = (y-1)^2$

SOLUTION: Setting $2y = (y-1)^2 - 1$ yields $0 = y^2 - 4y = y(y-4)$, so the two curves intersect at $y = 0$ and $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

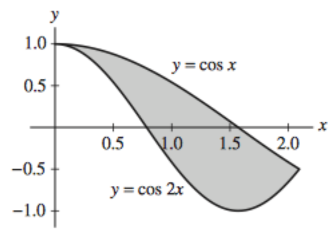
$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right)\,dy = \int_0^4 (4y - y^2)\,dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



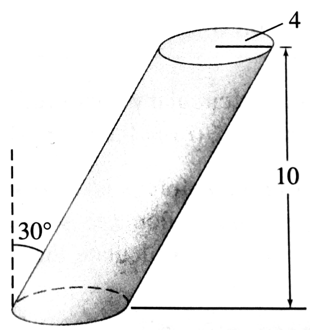
(e) $y = \cos x, y = \cos(2x), x = 0, x = \frac{2\pi}{3}$

SOLUTION: From the graph below, we see that $y = \cos x$ lies above $y = \cos 2x$ over the interval $[0, \frac{2\pi}{3}]$. The area of the region enclosed by the two curves is therefore

$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \frac{1}{2}\sin x\right)\Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.$$



(2) Calculate the volume of a cylinder inclined at an angle $\theta = \frac{\pi}{6}$ with height 10 and base of radius 4.



SOLUTION: By Cavalieri's Principle, the volume of this thing is the same as the volume of a regular cylinder of height 10. So the volume is $\pi R^2 h = \pi(4)^2(10) = 160\pi$.

Alternatively, the cross-sectional area is at each y-value $\pi(4)^2 = 16\pi$, so the volume is

$$\int_0^{10} 16\pi\, dy = 160\pi.$$

(3) Calculate the volume of the ramp in the figure below in three ways by integrating the area of the cross sections:

(a) perpendicular to the x-axis.

SOLUTION: Cross sections perpindicular to the x-axis are rectangles of width 4 and height $2 - \frac{1}{3}x$. The volume of the ramp is then

$$\int_0^6 4\left(-\frac{1}{3}x + 2\right)\, dx = \left(-\frac{2}{3}x^2 + 8x\right)\Big|_0^6 = 24.$$

(b) perpendicular to the y-axis.

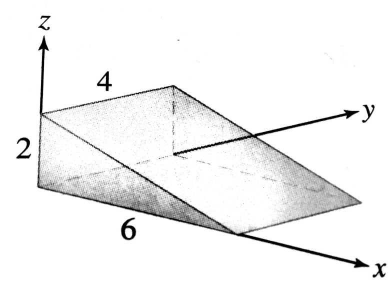
SOLUTION: Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2}\cdot 2 \cdot 6\right)\,dy = (6y)\Big|_0^4 = 24.$$

(c) perpendicular to the z-axis.

SOLUTION: Cross sections perpendicular to the z-axis are rectangles of length $6-3z$ and width 4. The volume of the ramp is then

$$\int_0^2 4(-3(z-2))\,dz = (-6z^2+24z)\Big|_0^2 = 24.$$



(4) Let M be the average value of $f(x) = 2x^2$ on $[0,2]$. Find a value c such that $f(c) = M$.

SOLUTION: First find the average value

$$M = \frac{1}{2-0}\int_0^2 2x^2\,dx = \frac{2}{2}\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}.$$

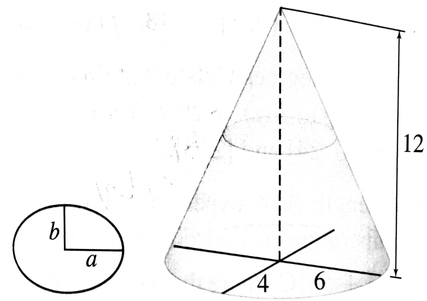
Then $M = f(c) = 2c^2 = \frac{8}{3}$ implies $c = \frac{2}{\sqrt{3}}$.

(5) Find the flow rate through a tube of radius 2 meters, if it's fluid velocity at distance r meters from the center is $v(r) = 4 - r^2$.

SOLUTION: The flow rate is (in meters cubed per second)

$$2\pi\int_0^R rv(r)\,dr = 2\pi\int_0^2 r(4-r^2)\,dr = 2\pi\left(2r^2 - \frac{1}{r}r^4\right)\Big|_0^2 = 8\pi.$$

(6) Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.



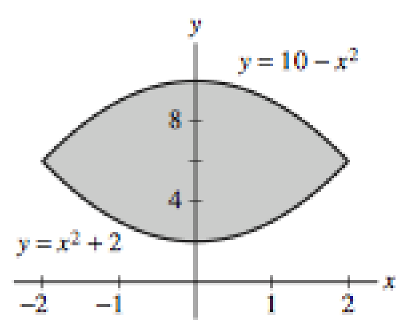
SOLUTION: At each height y, the elliptical cross section has major axis $\frac{1}{2}(12 - y)$ and minor axis $\frac{1}{3}(12 - y)$. The cross-sectional area is then $\frac{\pi}{6}(12 - y)^2$, and the volume is

$$\int_0^{12} \frac{\pi}{6}(12 - y)^2\, dy = -\frac{\pi}{18}(12 - y)^3\Big|_0^{12} = 96\pi.$$

(7) Sketch the region enclosed by the curves, and determine the cross section perpendicular to the x-axis. Set up an integral for the volume of revolution obtained by rotating the region around the x-axis, but do not evaluate.

(a) $y = x^2 + 2$, $y = 10 - x^2$.

SOLUTION: Setting $x^2 + 2x = 10 - x^2$ yields $x = \pm 2$. The region enclosed is the figure below.

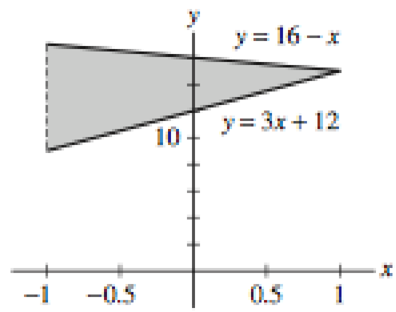


When this is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is then

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2\right)\, dx$$

(b) $y = 16 - x, y = 3x + 12, x = -1.$

SOLUTION: Setting $16 - x = 3x + 12$ gives $x = 1$. The region enclosed is in the picture below.



When rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$. So the volume of the solid of revolution is

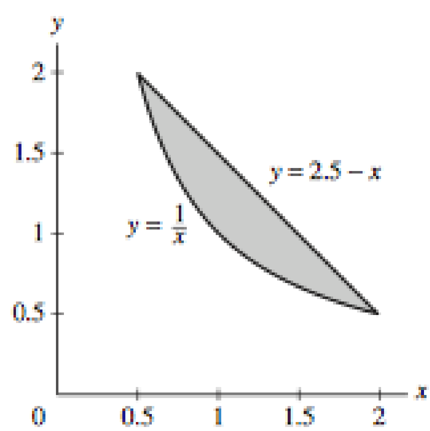
$$\pi \int_{-1}^{1} \left((16 - x)^2 - (3x + 12)^2 \right) \, dx.$$

(c) $y = \frac{1}{x}, y = \frac{5}{2} - x.$

SOLUTION: Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x - 2)(x - \tfrac{1}{2}).$$

So $x = 2$ and $x = \frac{1}{2}$ are the two points of intersection. The region enclosed by the curves is in the picture below.

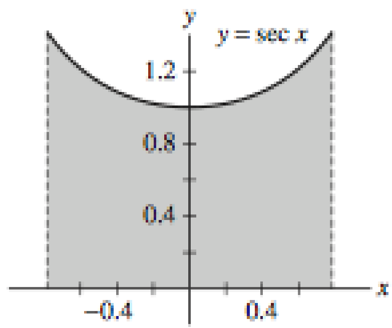


The cross sections are washers with outer radius $R = \frac{5}{2} - x$ and inner radius $r = \frac{1}{x}$. So the volume is

$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right) dx$$

(d) $y = \sec x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$.

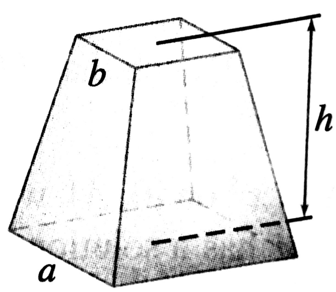
SOLUTION: The region in question is



When rotated around the x-axis, each cross section is a circular disk with radius $R = \sec x$. The volume of this solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2 \, dx$$

(8) A frustrum of a pyramid is a pyramid with it's top cut off. Let V be the volume of a frustrum of height h whose base is a square of side a and whose top is a square of side b with $a > b > 0$.



(a) Show that if the frustrum were continued to a full pyramid (i.e. the top wasn't cut off), it would have height $ha/(a-b)$.

SOLUTION: Let H be the height of the full pyramid. Using similar triangles, we have the proportion $\frac{H}{a} = \frac{H-h}{b}$, which gives $H = \frac{ha}{a-b}$.

(b) Show that the cross sectional area at height x is a square of side $(1/h)(a(h-x)+bx)$.

SOLUTION: Let w denote the side length of the square cross-section at height x. By similar triangles, we have $\frac{a}{H} = \frac{w}{H-x}$. Substituting H from part (a) gives

$$w = \frac{a(h-x)+bx}{h}.$$

(c) Show that $V = \frac{1}{3}h(a^2+ab+b^2)$.

SOLUTION: The volume of the frustrum is

$$\begin{aligned}\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2 dx &= \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right) dx \\ &= \frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\Bigg|_0^h \\ &= \frac{h}{3}\left(a^2+ab+b^2\right)\end{aligned}$$