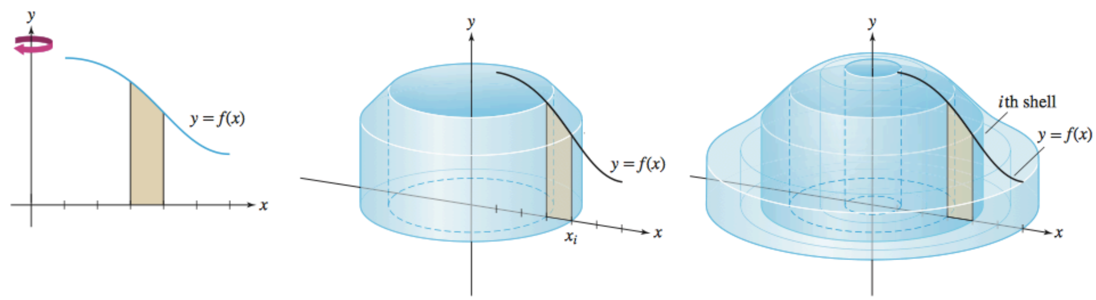
One-Page Review

§6.4 (Shell Method), §6.5 (Work and Energy)

MATH 1910 Recitation
September 27, 2016

(1) **Shell method:** When you rotate the region between two graphs around an axis, the segments **parallel** to the axis generate cylindrical shells. The volume V of the solid of revolution is the integral of the surface areas of shells.

$$V = \int \boxed{2\pi(\text{radius})(\text{height of shell})}^{(1)} .$$



(2) What is the volume of:

(a) The region between $f(x)$ and the x-axis ($f(x) \geq 0$) for $x \in [a, b]$ rotated around the y-axis?

$$V = \boxed{2\pi \int_a^b x f(x)\, dx.}^{(2)}$$

(b) The region between $f(x)$ and $g(x)$, ($f(x) \geq g(x) \geq 0$) for $x \in [a, b]$ rotated around the y-axis?

$$V = \boxed{2\pi \int_a^b (f(x) - g(x))\, dx.}^{(3)}$$

(c) The region between $f(x)$ and the x-axis ($f(x) \geq 0$) for $x \in [a, b]$, rotated around the line $x = c$ ($c \leq a$)? What if $c \geq a$?

If $c \leq a$, $V = \boxed{2\pi \int_a^b (x - c) f(x)\, dx.}^{(4)}$ If $c \geq a$, $V = \boxed{2\pi \int_a^b (c - x) f(x)\, dx.}^{(5)}$

(3) The work W performed to move an object from a to b along the x-axis by applying a force of magnitude $F(x)$ is $W = \boxed{\int_a^b F(x)\, dx}^{(6)}$.

(4) To compute work against gravity, first decompose an object into N layers of equal thickness Δy, and then express the work performed on a thin layer as $L(y)\Delta y$, where

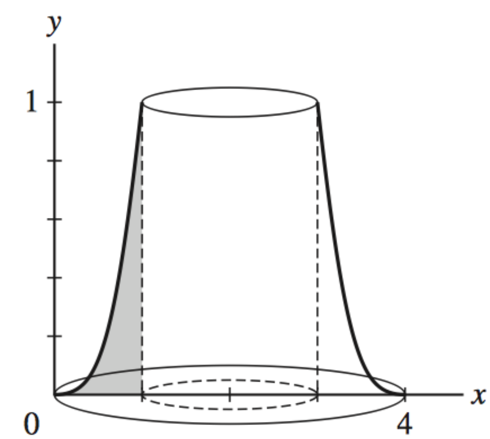
$$L(y) = g \times \text{density} \times \boxed{\text{area of } y}^{(7)} \times \boxed{\text{distance lifted}}^{(8)} .$$

Then the total work performed is $W = \boxed{\int_a^b L(y)\, dy}^{(9)}$.

(1) Sketch the solid obtained by rotating the region underneath the graph of f over the interval about the given axis, and calculate its volume using the shell method.

(a) $f(x) = x^3$, $x \in [0, 1]$, about $x = 2$.

SOLUTION:

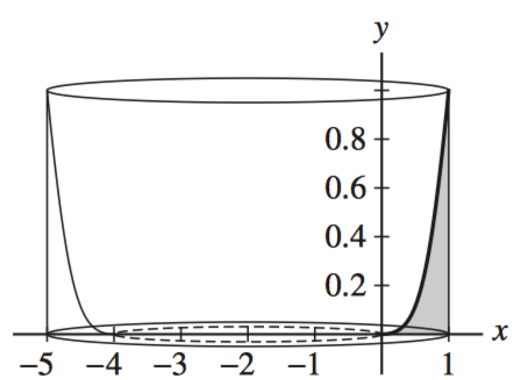


Each shell has radius $2 - x$ and height x^3, so the volume of this solid is

$$2\pi \int_0^1 (2-x)(x^3)\,dx = 2\pi \int_0^1 (2x^3 - x^4)\,dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{2} \bigg|_0^1 = \frac{3\pi}{5} \right).$$

(b) $f(x) = x^3$, $x \in [0, 1]$ about $x = -2$.

SOLUTION:

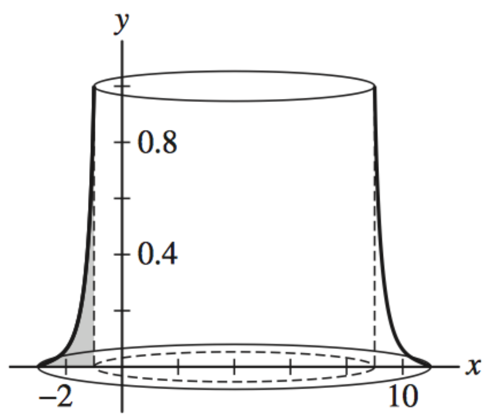


Each shell has radius $x - (-2) = x + 2$, and height x^3, so the volume of the solid is

$$2\pi \int_0^1 (2+x)x^3\, dx = 2\pi \int_0^1 (2x^3 + x^4)\, dx = 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{7\pi}{5}$$

(c) $f(x) = x^{-4}$, $x \in [-3, -1]$, about $x = 4$.

SOLUTION:

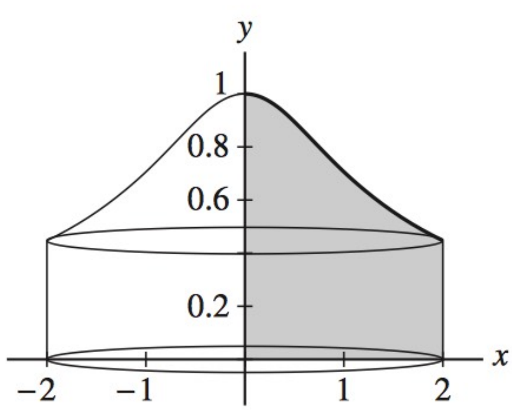


Each shell has radius $4 - x$, and height x^{-4}, so the volume of the solid is

$$2\pi \int_{-3}^{-1} (4-x)x^{-4}\, dx = 2\pi \int_{-3}^{-1} (4x^{-4} - x^{-3})\, dx = 2\pi \left(\frac{1}{2}x^{-2} - \frac{4}{3}x^{-3} \right) \Big|_{-3}^{-1} = \frac{280\pi}{81}.$$

(d) $f(x) = \frac{1}{\sqrt{x^2+1}}$, $x \in [0, 2]$, about $x = 4$.

SOLUTION:



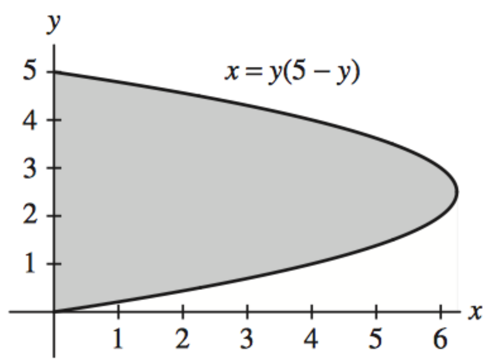
Each shell has radius x, and height $\frac{1}{\sqrt{x^2+1}}$, so the volume of the solid is

$$2\pi \int_0^2 x\left(\frac{1}{\sqrt{x^2+1}}\right)\,dx = 2\pi\sqrt{x^2+1}\Big|_0^2 = 2\pi(\sqrt{5}-1)$$

(2) Use the most convenient method (disk/washer or shell) to find the given volume of rotation.

(a) Region between $x = y(5-y)$ and $x = 0$, rotated around the y-axis.

SOLUTION: Examine the picture below, which shows the region in question. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of a slice is always along the parabola and the left endpoint is on the y-axis. So it's easier to do horizontal slices.

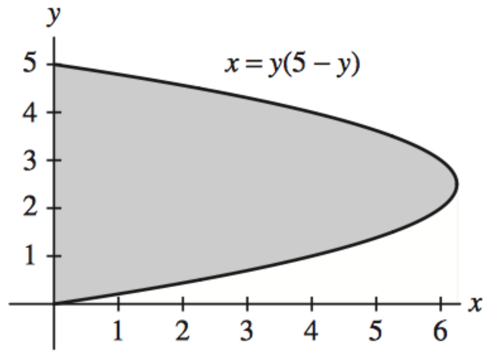


Now suppose the region is rotated about the y-axis. Because a horizontal slice is perpindicular to the y-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = y(5-y)$, so the volume is

$$\pi\int_0^5 y^2(5-y)^2\,dy = \pi\int_0^5 (25y^2 - 10y^3 + y^4)\,dy = \pi\left(\frac{25}{3}y^3 - \frac{5}{2}y^4 + \frac{1}{5}y^5\right)\Big|_0^5 = \frac{625\pi}{6}.$$

(b) Region between $x = y(5-y)$ and $x = 0$, rotated about the x-axis.

SOLUTION: Examine the figure below, which shows the region bounded by $x = y(5-y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5?y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

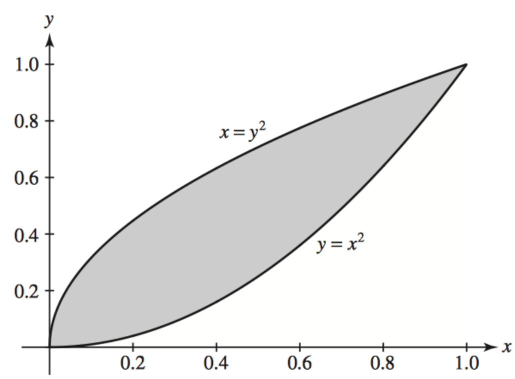


Now, suppose the region is rotated about the x-axis. Because a horizontal slice is parallel to the x-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of y and a height of $y(5?y)$, so the volume is

$$2\pi \int_0^5 y^2(5-y)\,dy = 2\pi \int_0^5 (5y^2 - y^3)\,dy = 2\pi \left(\frac{5}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^5 = \frac{625\pi}{6}.$$

(c) Region between $y = x^2$ and $x = y^2$, rotated about the y-axis.

SOLUTION: Examine the ?gure below, which shows the region bounded by $y = x^2$ and $x = y^2$. If the indicated region is sliced vertically, then the top of the slice lies along $x = y^2$ and the bottom lies along $y = x^2$. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along $y = x^2$ and left endpoint always lies along $x = y^2$. Thus, for this region, either choice of slice will be convenient. To proceed, let?s choose a vertical slice.

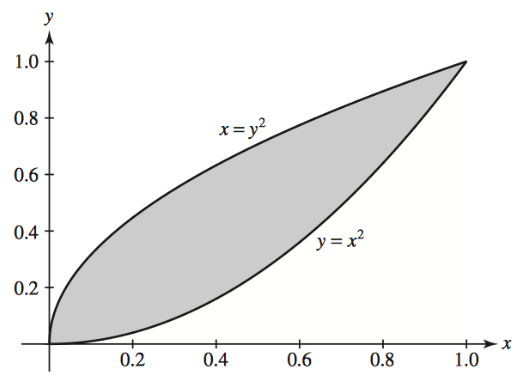


Now, suppose the region is rotated about the y-axis. Because a vertical slice is parallel to the y-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of x and a height of $\sqrt{x}?x^2$, so the volume is

$$2\pi \int_0^1 x(\sqrt{x} - x^2)\,dx = 2\pi \int_0^1 (x^{3/2} - x^3)\,dx = 2\pi \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{3\pi}{10}.$$

(d) Region between $y = x^2$ and $x = y^2$, rotated about $x = 3$.

SOLUTION: Examine the figure below, which shows the region bounded by $y = x^2$ and $x = y^2$. If the indicated region is sliced vertically, then the top of the slice lies along $x = y^2$ and the bottom lies along $y = x^2$. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along $y = x^2$ and left endpoint always lies along $x = y^2$. Thus, for this region, either choice of slice will be convenient. To proceed, let?s choose a vertical slice.



Now, suppose the region is rotated about $x = 3$. Because a vertical slice is parallel to $x = 3$, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of 3?x and a height of $\sqrt{x} = x^2$, so the volume is

$$2\pi \int_0^1 (3-x)(\sqrt{x}-x^2)\,dx = 2\pi \int_0^1 (3x^{1/2} - x^{3/2} - 3x^2 + x^3)\,dx$$
$$= 2\pi \left(2x^{3/2} - \frac{2}{5}x^{5/2} - x^3 + \frac{1}{4}x^4\right)\Bigg|_0^1 = \frac{17\pi}{10}.$$

(3) Calculate the work (in Joules) required to pump all of the water out of a full tank with the shape described. Distances are in meters, and the density of water is 1000 kg/m^3.

(a) A rectangular tank, with water exiting from a small hole at the top.

SOLUTION: Place the origin at the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)(32)\delta y = 313600\delta y \text{ N}.$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_0^5 313600y\,dy = 156800y^3\Bigg|_0^5 = 3.92 \times 10^6\text{J}.$$

(b) A horizontal cylinder of length ℓ, where water exits from a small hole at the top.

SOLUTION: Place the origin along the central axis of the cylinder. At location y, the layer of water is a rectangular slab of length ℓ, width $2\sqrt{r^2-y^2}$ and thickness δy. Thus, the volume of the layer is $2\ell\sqrt{r^2-y^2}\Delta y$ and the force needed to lift the layer is $19600\ell\sqrt{r^2-y^2}\delta y$. The layer must be lifted a distance $r-y$, so the total work needed to empty the tank is given by

$$\int_{-r}^{r} 19600\ell\sqrt{r^2-y^2}(r-y)\,dy = 19600\ell r\int_{-r}^{4}\sqrt{r^2-y^2}\,dy - 19600\int_{-r}^{2} y\sqrt{r^2-y^2}\,dy$$

Now

$$\int_{-r}^{r} y\sqrt{r^2-y^2}\,dy = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero (you could also see this by making the substitution $u = y^2$). Moreover, the other integral is one-half the area of a circle of radius r; thus,

$$\int_{-r}^{2}\sqrt{r^2-y^2}\,dy = \frac{1}{2}\pi r^2.$$

So the total work needed to empty the tank is

$$19600\ell r\left(\frac{1}{2}\pi r^2\right) - 19600\ell(0) = 9800\ell\pi r^3 \text{ Joules.}$$

(c) A trough as in the picture, where water exits by pouring over the sides.

SOLUTION: Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at height y meters is

$$w = a + \frac{y(b-a)}{h} \text{ meters,}$$

so the volume of each layer is

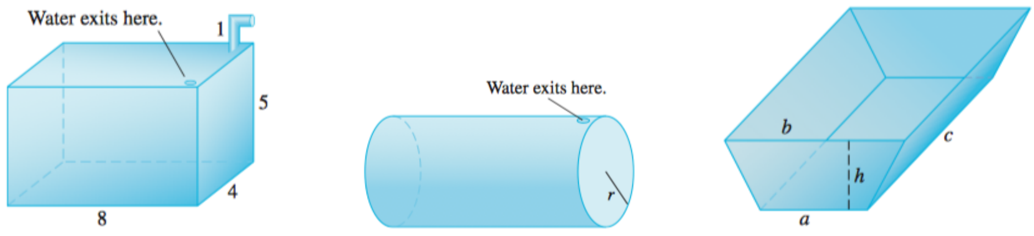
$$wc\Delta y = c\left(a + \frac{y(b-a)}{h}\right)\Delta y\text{meters}^3.$$

Thus, the force needed to lift a layer is

$$9800c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ Newtons.}$$

Each layer must be lifted $h-y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y(b-a)}{h}\right)dy = 9800c\left(\frac{ah^2}{3} + \frac{bh^2}{6}\right) \text{ Joules.}$$



(a) (b) (c)

(4) Calculate the work required to lift a 6 meter chain with mass 18 kg over the side of a building.

SOLUTION: First, note that the chain has a mass density of 3 kg/m. Now, consider a segment of the chain of length Δy located at distance y_j feet from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (3\Delta y)9.8 y_j \text{ Newtons.}$$

Summing over all segments of the chain and passing to the limit at $\Delta y \to 0$, it follows that the total work is

$$\int_0^6 29.4y\,dy = 14.7y^2\Big|_0^6 = 529.2 \text{ Joules.}$$

(5) A 3 meter chain with mass density $\rho(x) = 2x(4-x)$ kg/m lies on the ground. Calculate the work required to lift the chain from the front end so that its bottom is 2 meters above the ground.

SOLUTION: Consider a segment of the chain of length Δx that must be lifted x_j meters. The work needed to lift this segment is approximately

$$W_j \approx (\rho(x_j)\Delta x)9.8 x_j \text{ Joules.}$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work needed to fully extend the chain is

$$\int_0^3 9.8\rho(x)x\,dx = 9.8\int_0^3 (8x^2 - 2x^3)\,dx = 9.8\left(\frac{8}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^3 = 308.7 \text{ Joules.}$$

But we also need to lift the chain two meters off the ground after it's fully extended! This requires us to do work equal to 2 meters multiplied by the weight of the chain.

The weight of the chain is

$$\int_0^3 9.8\rho(x)\,dx = 9.8\int_0^3 (8x - 2x^2)\,dx = 9.8\left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 176.4 \text{ Newtons.}$$

So lifting it another two meters after it's fully extended requires an additional 352.8 Joules of work. The total work is therefore 661.5 Joules.