One-Page Review

MATH 1910 Recitation

§7.7 (L'Hôspital's Rule), §7.8 (Inverse Trig)

October 13, 2016

(1) **L'Hôspital's Rule:** If $f(a) = g(a) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \boxed{\lim_{x \to a} \frac{f'(x)}{g'(x)}}$ (1) .

(2) What are all the indeterminate forms? There are seven of them. $\boxed{\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \cdot \infty,\ \infty - \infty,\ 0^0,\ 1^\infty,\ \infty^0}$ (2)

(3) To evaluate the limit involving an indeterminate form 0^0, 1^∞, or ∞^0, first $\boxed{\text{take the logarithm}}$ (3) and then apply L'Hôspital's rule.

(4) Domain and range of inverse trigonometric functions.

(a) What is the domain of $\sin^{-1}$? $\boxed{[-1, 1]}$ (4) What is the range of $\sin^{-1}$? $\boxed{[-\pi/2, \pi/2]}$ (5)

(b) What is the domain of $\cos^{-1}$? $\boxed{[-1, 1]}$ (6) What is the range of $\cos^{-1}$? $\boxed{[0, \pi]}$ (7)

(c) What is the domain of $\tan^{-1}$? $\boxed{(-\infty, \infty)}$ (8) What is the range of $\tan^{-1}$? $\boxed{(-\pi/2, \pi/2)}$ (9)

(d) What is the domain of $\cot^{-1}$? $\boxed{(-\infty, \infty)}$ (10) What is the range of $\cot^{-1}$? $\boxed{(0, \pi)}$ (11)

(e) What is the domain of $\sec^{-1}$? $\boxed{(-\infty, -1) \cup (1, \infty)}$ (12) What is the range of $\sec^{-1}$? $\boxed{[0, \pi/2) \cup (\pi/2, \pi]}$ (13)

(f) What is the domain of $\csc^{-1}$? $\boxed{(-\infty, -1) \cup (1, \infty)}$ (14) What is the range of $\csc^{-1}$? $\boxed{[-\pi/2, 0) \cup (0, \pi/2]}$ (15)

(5) Derivatives of inverse trigonometric functions.

(a) $\frac{d}{dx} \sin^{-1}(x) = \boxed{\frac{1}{\sqrt{1-x^2}}}$ (16)

(b) $\frac{d}{dx} \tan^{-1}(x) = \boxed{\frac{1}{x^2+1}}$ (17)

(c) $\frac{d}{dx} \sec^{-1}(x) = \boxed{\frac{1}{|x|\sqrt{x^2-1}}}$ (18)

(d) $\frac{d}{dx} \cos^{-1}(x) = \boxed{\frac{-1}{\sqrt{1-x^2}}}$ (19)

(e) $\frac{d}{dx} \cot^{-1}(x) = \boxed{\frac{-1}{x^2+1}}$ (20)

(f) $\frac{d}{dx} \csc^{-1}(x) = \boxed{\frac{-1}{|x|\sqrt{x^2-1}}}$ (21)

(6) Integrals of inverse trigonometric functions.

(a) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \boxed{\sin^{-1}(x) + C}$ (22)

(b) $\int \frac{1}{x^2+1}\, dx = \boxed{\tan^{-1}(x) + C}$ (23)

(c) $\int \frac{1}{|x|\sqrt{x^2-1}} = \boxed{\sec^{-1}(x) + C}$ (24)

SOLUTIONS

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(1) Use L'Hôspital's Rule to calculate the limit

(a) $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x}$

SOLUTION:

$$\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x} = \lim_{x\to\infty} \frac{\frac{2}{3}x^{-1/3}+3}{\frac{5}{3}x^{2/3}-1} = \frac{0+3}{\infty-1} = 0.$$

(b) $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7}$

SOLUTION:

$$\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7} = \lim_{x\to\infty} \frac{9x^2+8x}{12x^2} = \lim_{x\to\infty}\left(\frac{9}{12}+\frac{8}{12x}\right) = \frac{3}{4}.$$

(c) $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2}$

SOLUTION: We actually need L'Hôspital's rule for this one! If you plug in $x=8$ you get the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2} = \lim_{x\to 8} \frac{\frac{5}{3}x^{2/3}-2}{\frac{1}{3}x^{-2/3}} = \lim_{x\to 8}(5x^{4/3}-6x^{2/3}) = 5(8)^{4/3}-6(8)^{2/3} = 56$$

(d) $\lim_{x\to 0} \frac{\tan 4x}{\tan 5x}$

SOLUTION:

$$\lim_{x\to 0} \frac{\tan 4x}{\tan 5x} = \lim_{x\to 0} \frac{\cos 5x}{\cos 4x}\cdot\frac{\sin 4x}{4x}\cdot\frac{5x}{\sin 5x} = \frac{4}{5}$$

(e) $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right)$

SOLUTION:

$$\begin{aligned}\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{x\cos x - \sin x}{x \sin x} \\ &= \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x} \\ &= \lim_{x\to 0} \frac{-x\sin x}{x\cos x + \sin x} \\ &= \lim_{x\to 0} \frac{-x\cos x - x}{-x\sin x + \cos x + \cos x} \\ &= \frac{0}{2} = 0\end{aligned}$$

(f) $\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x$

SOLUTION:

$$\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x=\lim_{x\to\pi/2}\frac{x-\pi/2}{1/\tan x}=\lim_{x\to\pi/2}\frac{x-\pi/2}{\cot x}=\lim_{x\to\pi/2}\frac{1}{-\csc^2(x)}=\lim_{x\to\pi/2}-\sin^2 x=-1$$

(g) $\lim_{x\to 0}\frac{x^2}{1-\cos x}$

SOLUTION:

$$\lim_{x\to 0}\frac{x^2}{1-\cos x}=\lim_{x\to 0}\frac{2x}{\sin x}=\lim_{x\to 0}\frac{2}{\cos x}=2$$

(h) $\lim_{x\to 0}\left(\frac{1}{x^2}-\csc^2 x\right)$

SOLUTION:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^2}-\csc^2 x\right)&=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\\ &=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\\ &=\lim_{x\to 0}\frac{2\sin x\cos x-2x}{2x^2\sin x\cos x+2x\sin^2 x}\\ &=\lim_{x\to 0}\frac{\sin 2x-2x}{x^2\sin 2x+2x\sin^2 x}\\ &=\lim_{x\to 0}\frac{2\cos 2x-2}{2x^2\cos 2x+2x\sin 2x+4x\sin x\cos x+2\sin^2 x}\\ &=\lim_{x\to 0}\frac{\cos 2x-1}{x^2\cos 2x+2x\sin 2x+\sin^2 x}\\ &=\lim_{x\to 0}\frac{-2\sin 2x}{-2x^2\sin 2x+2x\cos 2x+4x\cos 2x+2\sin 2x+2\sin x\cos x}\\ &=\lim_{x\to 0}\frac{-2\sin 2x}{(3-2x^2)\sin 2x+6x\cos 2x}\\ &=\lim_{x\to 0}\frac{-4\cos 2x}{2(3-2x^2)\cos 2x-4x\sin 2x+-12x\sin 2x+6\cos 2x}\\ &=-\frac{1}{3}\end{aligned}$$

(i) $\lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}$

SOLUTION:

$$\lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}=\lim_{x\to 2}\frac{2xe^{x^2}}{1}=4e^4$$

(j) $\displaystyle\lim_{x\to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x}$

SOLUTION:

$$\lim_{x\to 1} \frac{x(\ln x - 1)}{(x-1)\ln x} = \lim_{x\to 1} \frac{x\frac{1}{x} + (\ln x - 1)}{(x-1)(\frac{1}{x}) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}$$

(k) $\displaystyle\lim_{x\to\infty} \frac{e^x - e}{\ln x}$

SOLUTION:

$$\lim_{x\to\infty} \frac{e^x - e}{\ln x} = \lim_{x\to\infty} \frac{e^x}{x^{-1}} = \frac{e}{1} = e$$

(l) $\displaystyle\lim_{x\to\infty} \frac{e^{2x} - 1 - x}{x^2}$

SOLUTION:

$$\lim_{x\to\infty} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x\to\infty} \frac{2e^{2x} - 1}{2x} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$$

(m) $\displaystyle\lim_{x\to\infty} x^{1/x^2}$

SOLUTION: First, compute

$$\lim_{x\to\infty} \ln x^{1/x^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x^2} = \lim_{x\to\infty} e^{\ln x^{1/x^2}} = e^0 = 1.$$

(n) $\displaystyle\lim_{x\to 0^+} x^{\sin x}$

SOLUTION: First, compute

$$\begin{aligned}
\lim_{x\to 0^+} \ln x^{\sin x} &= \lim_{x\to 0^+} \sin x \ln x \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\cos x(\sin x)^{-2}} \\
&= \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x} \\
&= \lim_{x\to 0^+} -\frac{2\sin x\cos x}{-x\sin x + \cos x} = 0
\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\ln x^{\sin x}} = e^0 = 1.$$

(2) Find the derivative.

(a) $y = \arctan(x/3)$

SOLUTION: $y' = \dfrac{1}{(x^2/3)+3}$

(b) $y = \sec^{-1}(x+1)$

SOLUTION: $y' = \dfrac{1}{|x+1|\sqrt{x^2+2x}}$

(c) $y = e^{\cos^{-1}(x)}$

SOLUTION: $y' = \dfrac{-e^{\cos^{-1}(x)}}{\sqrt{1-x^2}}$

(d) $y = \csc^{-1}(x^{-1})$

SOLUTION: $y' = \dfrac{1}{\sqrt{1-x^2}}$

(e) $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

SOLUTION: $y' = \dfrac{1}{t^2+1}$

(f) $y = \frac{\cos^{-1}(x)}{\sin^{-1}(x)}$

SOLUTION: $\dfrac{-\pi}{2\sqrt{1-x^2}(\sin^{-1}(x))^2}$

(g) $y = \cos^{-1}(x + \sin^{-1}(x))$

SOLUTION: $y' = \dfrac{-1}{\sqrt{1-(x+\sin^{-1}x)^2}}\left(1+\dfrac{1}{\sqrt{1-x^2}}\right)$

(h) $y = \ln(\arcsin(x))$

SOLUTION: $y' = \dfrac{1}{\arcsin x\sqrt{1-x^2}}$

(3) Evaluate the integral

(a) $\displaystyle\int_0^4 \frac{1}{4x^2+9}\,dx$

SOLUTION: Let $x = (3/2)u$. Then $dx = (3/2)du$, and $4x^2+9 = 9u^2+9 = 9(u^2+1)$, and

$$\int_0^4 \frac{1}{4x^2+9}\,dx = \frac{1}{6}\int_0^{8/3}\frac{1}{u^2+1}\,du = \frac{1}{6}\tan^{-1}u\Big|_0^{8/3} = \frac{1}{6}\tan^{-1}\left(\frac{8}{3}\right)$$

(b) $\displaystyle\int_{-1/5}^{1/5} \frac{1}{\sqrt{4-25x^2}}\,dx$

SOLUTION: Let $x = 2u/5$. Then $dx = \frac{2}{5}du$, and $4-25x^2 = 4(1-u^2)$. So

$$\int_{-1/5}^{1/5} \frac{1}{\sqrt{4-25x^2}}\,dx = \frac{2}{5}\int_{-1/2}^{1/2} \frac{1}{\sqrt{4(1-u^2)}}\,du = \frac{1}{5}\sin^{-1} u\Big|_{-1/2}^{1/2} = \frac{\pi}{12}$$

(c) $\displaystyle\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2-1}}\,dx$

SOLUTION: Let $x = u/4$. Then $dx = du/4$, $16x^2-1 = u^2-1$, and

$$\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2-1}}\,dx = \int_{\sqrt{2}}^{2} \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1} u\Big|_{\sqrt{2}}^{2} = \frac{\pi}{12}$$

(d) $\displaystyle\int \frac{1}{x\sqrt{x^4-1}}\,dx$

SOLUTION: Let $u = x^2$. Then $du = 2x\,dx$, and

$$\int \frac{1}{x\sqrt{x^4-1}} = \int \frac{1}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1} u + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

(e) $\displaystyle\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx$

SOLUTION: Observe that

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

In the first integral on the right hand side, we let $u = 1-x^2$, $du = -2x\,dx$. Then

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du + \frac{1}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

(f) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

SOLUTION: Let $u = \tan^{-1}(x)$. Then $du = \frac{dx}{1+x^2}$, and

$$\int \frac{\tan^{-1}(x)}{1+x^2}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{(\tan^{-1} x)^2}{2} + C.$$

(g) $\int \frac{1}{\sqrt{5^{2x}1}}\,dx$

SOLUTION: First, rewrite

$$\int \frac{1}{\sqrt{5^{2x}1}}\,dx = \int \frac{1}{5^x\sqrt{1-5^{-2x}}} = \int \frac{5^{-x}}{\sqrt{1-5^{-2x}}}$$

Now let $u = 5^{-x}$. Then $du = -5^{-x}\ln 5\,dx$, and

$$\int \frac{1}{\sqrt{5^{2x}-1}} = -\frac{1}{\ln 5}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{\ln 5}\sin^{-1} u + C = -\frac{1}{\ln 5}\sin^{-1}(5^{-x}) + C$$