ONE-PAGE REVIEW

MATH 1910 Recitation

§7.9 (Hyperbolic Trig), §8.1 (Integration by Parts)

October 18, 2016

(1) $\sinh(x) = \boxed{\dfrac{e^x - e^{-x}}{2}}$ (1) $\qquad \cosh(x) = \boxed{\dfrac{e^x + e^{-x}}{2}}$ (2) $\qquad \tanh(x) = \boxed{\dfrac{\sinh(x)}{\cosh(x)}}$ (3)

$\coth(x) = \boxed{\dfrac{\cosh(x)}{\sinh(x)}}$ (4) $\qquad \operatorname{sech}(x) = \boxed{\dfrac{1}{\cosh(x)}}$ (5) $\qquad \operatorname{csch}(x) = \boxed{\dfrac{1}{\sinh(x)}}$ (6)

(2) Derivatives of hyperbolic trigonometric functions

$\dfrac{d}{dx}\sinh(x) = \boxed{\cosh(x)}$ (7) $\qquad \dfrac{d}{dx}\cosh(x) = \boxed{\sinh(x)}$ (8)

$\dfrac{d}{dx}\tanh(x) = \boxed{\operatorname{sech}^2(x)}$ (9) $\qquad \dfrac{d}{dx}\coth(x) = \boxed{-\operatorname{csch}^2(x)}$ (10)

$\dfrac{d}{dx}\operatorname{sech}(x) = \boxed{-\operatorname{sech}(x)\tanh(x)}$ (11) $\qquad \dfrac{d}{dx}\operatorname{csch}(x) = \boxed{-\operatorname{csch}(x)\coth(x)}$ (12)

(3) Integrals of hyperbolic trigonometric functions

$\int \sinh(x)\,dx = \boxed{\cosh(x) + C}$ (13) $\qquad \int \cosh(x)\,dx = \boxed{\sinh(x) + C}$ (14)

$\int \operatorname{sech}^2(x)\,dx = \boxed{\tanh(x) + C}$ (15) $\qquad \int \operatorname{csch}^2(x)\,dx = \boxed{-\coth(x) + C}$ (16)

$\int \operatorname{sech}(x)\tanh(x)\,dx = \boxed{-\operatorname{sech}(x) + C}$ (17) $\qquad \int \operatorname{csch}(x)\coth(x)\,dx = \boxed{-\operatorname{csch}(x) + C}$ (18)

(4) Inverse Hyperbolic Functions

Function	Domain	Derivative
$\sinh^{-1}(x)$	$(-\infty, \infty)$ (19)	$\dfrac{1}{\sqrt{x^2+1}}$ (20)
$\cosh^{-1}(x)$	$x \geq 1$ (21)	$\dfrac{1}{\sqrt{x^2-1}}$ (22)
$\tanh^{-1}(x)$	$\|x\| < 1$ (23)	$\dfrac{1}{1-x^2}$ (24)
$\coth^{-1}(x)$	$\|x\| > 1$ (25)	$\dfrac{1}{1-x^2}$ (26)
$\operatorname{sech}^{-1}(x)$	$0 < x \leq 1$ (27)	$\dfrac{-1}{x\sqrt{1-x^2}}$ (28)
$\operatorname{csch}^{-1}(x)$	$x \neq 0$ (29)	$\dfrac{-1}{\|x\|\sqrt{x^2+1}}$ (30)

(5) **Integration by parts**

$$\int u\,dv = \boxed{vu - \int v\,du} \quad (31)$$

SOLUTIONS

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(1) Simplify $\sinh(\ln x)$ and $\tanh(\frac{1}{2}\ln(x))$.

SOLUTION: $\sinh(\ln x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ and $\tanh(\dfrac{1}{2}\ln x) = \dfrac{x-1}{x+1}$ for $x > 0$.

(2) Find the derivative.

(a) $y = \ln(\cosh(x))$.

SOLUTION: $y' = \tanh(x)$

(b) $y = \operatorname{sech}(x)\coth(x)$.

SOLUTION: $y' = \operatorname{sech}(x)(-\operatorname{csch}^2(x) - 1)$

(3) Calculate the integral.

(a) $\displaystyle\int \cosh(2x)\,dx$

SOLUTION: $\dfrac{1}{2}\sinh(2x) + C$

(b) $\displaystyle\int \tanh(3t)\operatorname{sech}(3t)\,dt$

SOLUTION: $-\dfrac{1}{3}\operatorname{sech}(3t) + C$

(c) $\displaystyle\int \frac{\cosh(x)}{3\sinh(x) + 4}$

SOLUTION: $\dfrac{1}{3}\ln|3\sinh(x) + 4| + C$

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4}}$

SOLUTION: $\ln|x + \sqrt{x^2 - 4}| + C$

(e) $\displaystyle\int \frac{-1}{x\sqrt{x^2 + 16}}\,dx$

SOLUTION: Let $x = 4u$. Then $dx = 4du$ and

$$\int \frac{-1}{x\sqrt{x^2 + 16}}\,dx = -\frac{1}{4}\int \frac{1}{u\sqrt{u^2 + 1}}\,du = -\frac{1}{4}\operatorname{csch}^{-1}(u) + C = -\frac{1}{4}\operatorname{csch}^{-1}(\sqrt{x^2 + 16}) + C.$$

(NOTE: The answer given in the back of the book (below) is also correct, but it's very hard to see that this is equal to the answer given above. To get the answer that the book gave, use partial fractions.)

$$\frac{1}{4}\left(\ln|x| - \ln|4 + \sqrt{x^2+16}|\right) + C$$

(f) $\int xe^{-x}\,dx$

SOLUTION: Let $u = x$ and $dv = e^{-x}$. Then $u = x$, $du = dx$, and $v = -e^{-x}$. So

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int (1)(-e^{-x})\,dx = -e^{-x}(x+1) + C.$$

(g) $\int x^3 e^{x^2}\,dx$.

SOLUTION: Let $w = x^2$. Then $dw = 2x\,dx$ and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw.$$

Now use integration by parts with $u = w$ and $dv = e^w$. We have $du = 1$ and $v = e^w$, so

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw = we^w - \int (1)e^w\,dw = we^w - e^2.$$

Finally, substitute back $w = x^2$ to get

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

(h) $\int_1^3 \ln x\,dx$.

SOLUTION: Let $u = \ln x$ and $dv = 1$. Then $v = x$ and $du = 1/x$. So using integration by parts,

$$\int_1^3 \ln x\,dx = x\ln x\Big|_1^3 - \int_1^3 1\,dx = 3\ln 3 - 2.$$

(4) Find the volume of the solid obtained by revolving $y = \cos x$ for $0 \le x \le \pi/2$ around the y-axis.

SOLUTION: Using the cylindrical shells method, the volume V is given by

$$V = \int_a^b (2\pi r)h\,dx = 2\pi\int_0^{\pi/2} x\cos x\,dx.$$

and the radius $r = x$ varies from 0 to $\pi/2$, the height is $h = y = \cos x$. Then using integration by parts, with $u = x$ and $dv = \cos x$, we get

$$V = 2\pi \int_0^{\pi/2} x \cos x \, dx = 2\pi \left(x \sin x + \cos x \right) \Big|_0^{\pi/2} = \pi(\pi - 2).$$