ONE-PAGE REVIEW

MATH 1910 Recitation

§8.7 (Improper Integrals)
§8.8 (Probability and Integration)
§8.9 (Numerical Integration)

November 1, 2016

(1) The **improper integral** of f over $[a, \infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \boxed{\lim_{R\to\infty} \int_a^R f(x)\,dx.}^{(1)}$$

We say that the improper integral **converges** if [the limit exists](2), and **diverges** if [the limit does not exist](3).

(2) The p**-integral** is $\int_a^\infty \frac{1}{x^p}\,dx$. If [$p > 1$](4), then this integral converges. If [$p \le 1$](5), then it diverges.

(3) **Comparison test:** Assume $f(x) \ge g(x)$
If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ [converges](6).
If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ [diverges](7).

(4) If $p(x)$ is a **probability density function** or **PDF**, then $\int_{-\infty}^\infty p(x)\,dx =$ [1](8)

(5) If X is a random variable with probability density function p, then the probability that X is between a and b is

$$P(a \le X \le b) = \boxed{\int_a^b p(x)\,dx}^{(9)}$$

(6) The **mean** or **average value** of a random variable X with PDF $p(x)$ is [$\int_{-\infty}^\infty x p(x)\,dx$](10).

(7) There are three numerical approximations to $\int_a^b f(x)\,dx$:

(a) The **midpoint rule:** $M_N =$ [$\Delta x\,(f(c_1) + \ldots + f(c_N))\, right),$](11) $c_j = a + \left(j + \frac{1}{2}\right)\Delta x.$

(b) The **trapezoid rule:** $T_N =$ [$\frac{1}{2}\Delta x\,(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N)$](12)

(c) **Simpson's rule:** $S_N =$ [$\frac{1}{3}\Delta x\,(y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N)$](13)

SOLUTIONS

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(1) Determine whether the improper integral converges, and if it does, evaluate it.

(a) $\displaystyle\int_1^\infty \frac{1}{x^{20/19}}\,dx$

SOLUTION:

$$\begin{aligned}\int_1^\infty \frac{1}{x^{20/19}}\,dx &= \lim_{a\to\infty}\int_1^a \frac{1}{x^{20/19}}\,dx \\ &= \lim_{a\to\infty}\left(-19x^{-1/19}\right)\Big|_1^a \\ &= \lim_{a\to\infty}\left(-19-\frac{19}{a^{1/19}}\right) \\ &= 19-0=\boxed{19}\end{aligned}$$

(b) $\displaystyle\int_{20}^\infty \frac{1}{t}\,dt$

SOLUTION: The integral doesn't converge, because it's a p-integral with $p=1$.

(c) $\displaystyle\int_0^5 \frac{1}{x^{19/20}}\,dx$

SOLUTION: The function $x^{-19/20}$ is infinite at the endpoint zero, so it is improper.

$$\begin{aligned}\int_0^5 \frac{1}{x^{19/20}}\,dx &= \lim_{a\to 0}\int_a^5 \frac{1}{x^{19/20}} \\ &= \lim_{a\to 0}\left(20x^{1/20}\right)\Big|_a^5 \\ &= \lim_{a\to 0}\left(205^{1/20}-20a^{1/20}\right) \\ &= 20(5^{1/20}-0)=\boxed{20\cdot 5^{1/20}}\end{aligned}$$

(d) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$

SOLUTION: The function $f(x) = \frac{1}{\sqrt{3-x}}$ is infinite at $x = 3$, so it is improper.

$$\begin{aligned}\int_1^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{a\to 3}\int_1^a \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{a\to 3}\left(2\sqrt{3-x}\right)\Big|_1^a \\ &= \lim_{a\to 3} 2\sqrt{3-a} - 2\sqrt{2} \\ &= 2\sqrt{0} - 2\sqrt{2} = \boxed{2\sqrt{2}}\end{aligned}$$

(e) $\displaystyle\int_{-2}^4 \frac{1}{(x+2)^{1/3}}\,dx$

SOLUTION: The function $f(x) = (x+2)^{-1/3}$ is infinite at $x = -2$, so it is improper.

$$\begin{aligned}\int_{-2}^4 \frac{1}{(x+2)^{1/3}}\,dx &= \lim_{a\to -2}\int_a^4 \frac{1}{(x+2)^{1/3}}\,dx \\ &= \lim_{a\to 2}\frac{3}{2}(x+2)^{2/3}\Big|_a^4 \\ &= \lim_{a\to 2}\frac{3}{2}\left(6^{3/2} - (a+2)^{3/2}\right) \\ &= \frac{3}{2}\left(6^{2/3} - 0\right) = \boxed{\frac{3}{2}6^{2/3}}\end{aligned}$$

(2) Find a constant C such that $p(x) = {}^{C}/_{(2+x)^3}$ is a probability density function on the interval $[2, 4]$.

SOLUTION: For $p(x)$ to be a probability density function, it must integrate to 1 over the given integral. So

$$1 = \int_2^4 p(x)\,dx = \int_2^4 \frac{C}{(2+x)^3}\,dx = \frac{-C}{2(2+x)^2}\Big|_2^4 = C\left(\frac{-1}{72} + \frac{1}{32}\right) = C\frac{5}{288}$$

Therefore, $\boxed{C = \frac{288}{5}}$.

(3) A company produces boxes of rice that are filled on average with 16 oz of rice. Due to machine error, the actual volume of rice is normally distributed with a standard deviation of 0.4 oz. Find $P(X > 17)$, the probability of a box having more than 17 oz of rice.

SOLUTION: The problem tells us that the mean is $\mu = 16$ and the standard deviation of $\sigma = 0.4$. We need to find $P(X > 17)$. We have

$$\begin{aligned} P(X > 17) &= 1 - P(X \leq 17) \\ &= 1 - F\left(\frac{17-16}{0.4}\right) \\ &= 1 - F(2.5) \\ &= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.5} e^{-t^2/2}\, dt \\ &\approx 0.00621 \end{aligned}$$

(4) Find the T_4 approximation for $\int_0^4 \sqrt{x}\, dx$.

SOLUTION: Let $f(x) = \sqrt{x}$. We divide $[0, 4]$ into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1,$$

with endpoints $0, 1, 2, 3, 4$. With this data, we get

$$T_4 = \frac{1}{2}\Delta x \left(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}\right) \boxed{\approx 5.14626.}$$

(5) State whether M_{10} underestimates or overestimates $\int_1^4 \ln(x)\, dx$ and find a bound for the maximum possible error.

SOLUTION: Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on the interval $[1, 4]$, so $f(x)$ is concave down. Therefore, the midpoint rule overestimates the integral. Now use the error formula for M_{10}.

$$M_{10} \leq \frac{K_2(b-a)^3}{24N^2} = \frac{(1)(4-1)^3}{24(10)^2} \boxed{= 0.01125.}$$

To find K_2, notice that $|f''(x)| = |-1/x^2|$ has maximum value on $[1, 4]$ at $x = 1$, so we can take $K_2 = |-1/1^2| = 1$.