One-Page Review

MATH 1910 Recitation

§9.1 (Arc Length and Surface Area)
§9.4 (Taylor Polynomials)
§10.1 (Differential Equations)

November 3, 2016

(1) The **arc length** of $f(x)$ on the interval $[a,b]$ is $\boxed{\int_a^b \sqrt{1+f'(x)^2}\,dx.}$ (1)

(2) The **surface area** of the surface obtained by rotating the graph of $f(x)$ around the x-axis for $a \le x \le b$ is $\boxed{2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx.}$ (2)

(3) The **n-th Taylor Polynomial** centered at $x = a$ for the function f is

$$T_n(x) = \boxed{f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$ (3)

(4) The **error for the n-th Taylor Polynomial** is

$$|T_n(x) - f(x)| \le \boxed{K\frac{|x-a|^{n+1}}{(n+1)!}}\ .$$ (4)

(5) **Taylor's Theorem** says that

$$R_n(x) = T_n(x) - f(x) = \boxed{\frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,du.}$$ (5)

(6) A **differential equation** is like a normal equation, except you solve a differential equation for a [function] (6) instead of a number.

(7) The **order** of a differential equation is the highest derivative of y appearing in the equation. What are the orders of the following equations?

	Equation	Order
(a)	$y' = x^2$	1 (7)
(d)	$y''' + x^4 y' = 2$	3 (8)
(b)	$(y')^3 + yy' = \sin x$	1 (9)
(c)	$y'' = y^2$	2 (10)

(8) The technique for solving a differential equation where you move all the x-terms to one side and all of the y-terms to the other side is called [Separation of Variables.] (11)

SOLUTIONS

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§9.1 (Arc Length and Surface Area)

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§10.1 (Differential Equations)

(1) For the curve curve $y = \ln(\cos x)$ over the interval $[0, {}^{\pi}/_{4}]$, set up an integral to calculate:

(a) the arc length.

SOLUTION: First, calculate

$$1 + (y')^2 = 1 + \tan^2(x) = \sec^2(x),$$

so the arc length is

$$\int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/4} \sec(x)\, dx = \ln|\sec(x) + \tan(x)|\Big|_0^{\pi/4} = \boxed{\ln|\sqrt{2} + 1|}$$

(b) the surface area when rotated around the x-axis.

SOLUTION: As in the previous part, we have

$$1 + (y')^2 = \sec^2(x)$$

Therefore, plug into the arc length formula

$$\text{Surface Area } = 2\pi \int_0^{\pi/4} y\sqrt{1 + (y')^2} = \boxed{2\pi \int_0^{\pi/4} \ln(\cos(x))\sec(x)\, dx}$$

(2) Approximate the arc length of the curve $y = \sin(x)$ over the interval $[0, {}^{\pi}/_{2}]$ using the midpoint rule M_8.

SOLUTION: Since $y = \sin(x)$, we have

$$1 + (y')^2 = 1 + \cos^2(x)$$

Therefore, $\sqrt{1 + (y')^2} = \sqrt{1 + \cos^2(x)}$, and the arc length over $[0, \pi/2]$ is

$$\int_0^{\pi/2} \sqrt{1 + \cos^2(x)}\, dx.$$

Let $f(x) = \sqrt{1 + \cos^2(x)}$. M_8 is the midpoint approximation with eight subdivisions. So

$$\begin{aligned}
\Delta x &= \frac{\pi/2 - 0}{8} = \frac{\pi}{16} \\
x_i &= 0 + (i - \tfrac{1}{2})\Delta x \qquad \text{for } i = 1, 2, \ldots, 8 \\
y_i &= f\left((i - \tfrac{1}{2})\Delta x\right) \\
M_8 &= \sum_{i=1}^{8} y_i \Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_8)\Delta x
\end{aligned}$$

i	x_i	$f(x_i) = y_i$
1	0.5	1.41081
2	1.5	1.3841
3	2.5	1.3333
4	3.5	1.26394
5	4.5	1.18425
6	5.5	1.10554
7	6.5	1.04128
8	7.5	1.00479

The final answer is that the arc length is approximately $\boxed{1.9101}$.

(3) Find the Taylor polynomials $T_2(x)$ and $T_3(x)$ for $f(x) = \frac{1}{1+x}$ centered at $a = 1$.

SOLUTION: We need to take a few derivatives, and then plug in $a = 1$ to each one.

n	n-th derivative $f^{(n)}(x)$	$f^{(n)}(a)$
0	$f(x) = \dfrac{1}{1+x}$	$f(1) = 1/2$
1	$f'(x) = \dfrac{-1}{(1+x)^2}$	$f'(1) = -1/4$
2	$f''(x) = \dfrac{2}{(1+x)^3}$	$f''(1) = 1/4$
3	$f'''(x) = \dfrac{-6}{(1+x)^4}$	$f'''(1) = -3/8$

Then plug these values into the formula for the Taylor polynomial.

$$T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}$$

$$T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$$

(4) Find n such that $|T_n(1.3) - \sqrt{1.3}| \leq 10^{-6}$, where T_n is the Taylor polynomial for $\sqrt{x}$ at $a = 1$.

SOLUTION: By the error formula, we have that

$$|T_n(1.3) - \sqrt{1.3}| \leq \frac{K_{n+1}(1.3-1)^{n+1}}{(n+1)!}$$

So we just need to find n such that

$$\frac{K_{n+1}(0.3)^{n+1}}{(n+1)!} < 10^{-6},$$

where K_{n+1} is the maximum value of the $(n+1)$-st derivative of $f(x) = \sqrt{x}$ between 1 and 1.3. Since $f^{(n+1)}(x)$ is the $(n+1)$-st derivative of $\sqrt{x}$, and this always has x in the denominator for any $n \geq 0$, this maximum will always occur at $x = 1$. Therefore, in this case,

$$K_{n+1} = |f^{(n+1)}(1)|.$$

So we just need to find n such that

$$\frac{|f^{(n+1)}(1)|(0.3)^{n+1}}{(n+1)!} < 10^{-6}.$$

The hard part is finding a pattern for the n-th derivative of $\sqrt{x}$, but that's not strictly necessary, although possible. If you keep taking derivatives of $\sqrt{x}$ and plugging into the formula, you find that this is valid for $\boxed{n \geq 7}$.

Alternatively, the general formula for the n-th derivative of $\sqrt{x}$ is

$$f^{(n)}(x) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{\frac{-(2n-1)}{2}}$$

Then you can plug this in to the previous formula.

(5) Find the general solutions of the following differential equations using separation of variables.

(a) $\dfrac{dy}{dt} - 2y = 1$

SOLUTION: First, separate the variables:

$$\frac{dy}{1+2y} = dt$$

Then integrate both sides

$$\int \frac{dy}{1+2y} = \int dt$$

$$\Longrightarrow \frac{1}{2} \ln |1+2y| = t + C$$

$$\boxed{y = -\frac{1}{2} + Ce^{2t}}$$

(b) $(1+x^2)y' = x^3 y$

SOLUTION: First, separate the variables:

$$(1+x^2)\frac{dy}{dx} = x^3 y \Longrightarrow \frac{dy}{y} = \frac{x^3\, dx}{1+x^2}$$

Then integrate both sides

$$\int \frac{dy}{y} = \int \frac{x^3\, dx}{1+x^2}$$

Do polynomial long division to the right hand side.

$$\implies \ln|y| = \int x + \frac{-x}{1+x^2}\,dx = \frac{x^2}{2} - \frac{\ln|x^2+1|}{2} + C$$

Clear the logarithms, and absorb constants.

$$\boxed{y = \frac{Ce^{x^2/2}}{1+x^2}}$$

(6) Solve the initial value problem $\begin{cases} y' + 2y = 0 \\ y(\ln(2)) = 3 \end{cases}$

SOLUTION: First, separate variables

$$\frac{dy}{dx} = -2y \implies \frac{dy}{y} = -2\,dx.$$

Then integrate both sides

$$\int \frac{dy}{y} = \int -2\,dx \implies \ln|y| = -2x + C.$$

Now clear the natural logs by exponentiating.

$$y = Ce^{-2x}$$

Then plug in the initial value $y(\ln(2)) = 3$ to get

$$3 = Ce^{-2\ln(2)} \implies 3 = \frac{C}{4} \implies C = 12.$$

So the final answer is

$$\boxed{y = 12e^{-2x}}$$