ONE-PAGE REVIEW

§11.6 (Power Series)
§11.7 (Taylor Series)

MATH 1910 Recitation
November 22, 2016

(1) An infinite series of the form $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is called a [power series] (1) and c is called the [center] (2).

(2) The [radius of convergence] (3) of $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a constant R such that $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$. If $F(x)$ converges for all x, then $R =$ [∞] (4).

(3) To determine R, use [the ratio test.] (5)

(4) $\sum_{n=0}^{\infty} x^n =$ [$\dfrac{1}{1-x}$] (6), with $R =$ [1] (7).

(5) The powerseries $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the [Taylor Series] (8) for $f(x)$. If $c = 0$, this is called a [Maclaurin series] (9).

(6) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} =$ [e^x] (10)

(7) $(1+x)^a = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n$ for $|x| < 1$, where $\binom{a}{n} =$ [$\dfrac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$] (11)

SOLUTIONS

§11.6 (Power Series)
§11.7 (Taylor Series)

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(1) Show that all three of the following power series have the same radius of convergence, but different behavior at the endpoints.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$

SOLUTION: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} 9^n}{|x-5|^n 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.

But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$$x = 14: \qquad \sum_{n=1}^{\infty} \frac{(14-5)^n}{9^n} = \sum_{n=1}^{\infty} 1 \qquad \text{diverges}$$

$$x = -4: \qquad \sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n \qquad \text{diverges}$$

So the interval of convergence is $(-4, 14)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n9^n}$

SOLUTION: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} n 9^n}{|x-5|^n (n+1) 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.

But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$$x = 14: \qquad \sum_{n=1}^{\infty} \frac{(14-5)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges}$$

$$x = -4: \qquad \sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{converges}$$

The interval of convergence is $[-4, 14)$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 9^n}$

SOLUTION: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} n^2 9^n}{|x-5|^n (n+1)^2 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.
But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$$x = 14 : \qquad \sum_{n=1}^{\infty} \frac{(14-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{converges}$$

$$x = -4 : \qquad \sum_{n=1}^{\infty} \frac{(-4-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{converges}$$

The interval of convergence is $[-4, 14]$.

(2) Use the geometric series formula to expand the function $\dfrac{1}{1+3x}$ in a power series with center $c = 0$ and determine radius of convergence.

SOLUTION: The formula for the geometric series implies that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$. Replace x by $-3x$ in that formula to get

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^n.$$

This formula is valid for $|-3x| < 1$, or $|x| < 1/3$. So the radius of convergence is $R = \frac{1}{3}$.

(3) Write out the first four terms of the Taylor series $f(x)$ centered at $c = 3$ if $f(3) = 1$, $f'(3) = 2$, $f''(3) = 12$, $f'''(3) = 3$.

SOLUTION:

$$\begin{aligned} f(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \ldots \\ &= 1 + 2(x-3) + \frac{12}{2!}(x-3)^2 + \frac{3}{3!}(x-3)^3 + \ldots \\ &= 1 + 2(x-3) + 6(x-3)^2 + \frac{1}{2}(x-3)^3 + \ldots \end{aligned}$$

(4) Find the Taylor series of the following functions and determine the radius of convergence.

(a) $f(x) = \sin(2x)$, centered at $x = 0$.

SOLUTION:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

Since the formula for $\sin(x)$ is valid for all x, the formula for $\sin(2x)$ is also valid for all x.

(b) $f(x) = e^{4x}$, centered at $x = 0$.

SOLUTION:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

Since the formula for e^x is valid for all x, so is the formula for e^{4x}.

(c) $f(x) = x^2 e^{x^2}$, centered at $x = 0$.

SOLUTION:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$
$$x^2 e^{x^2} = x^2 \left(\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

Since the formula for x^2 is valid for all x, so is the formula for $x^2 e^{x^2}$.

(d) $f(x) = \dfrac{1}{3x-2}$, centered at $c = -1$.

SOLUTION: Rewrite the function as follows:

$$\frac{1}{3x-2} = \frac{1}{-5+3(x+1)} = \frac{-1}{5} \frac{1}{1 - \frac{3(x+1)}{5}}$$

Now use the geometric series formula, valid for $|x| < 1$.

$$\frac{1}{3x-2} = -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{3(x+1)}{5} \right)^n = -\frac{1}{5} \sum_{n=0}^{\infty} 3^n 5^n (x+1)^n = -\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} (x+1)^n$$

This formula is now valid for $\left| \frac{3(x+1)}{5} \right| < 1$, or $|x+1| < \frac{5}{3}$. So the radius of convergence is $\frac{5}{3}$.

(e) $f(x) = (1+x)^{1/3}$, centered at $c = 0$.

SOLUTION: Use the binomial series formula with $a = \frac{1}{3}$.

$$(1+x)^{\frac{1}{3}} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{3}}{n} x^n$$

The radius of convergence is 1, since the formula is valid for $|x| < 1$.

(f) $f(x) = \sqrt{x}$, centered at $c = 4$.

SOLUTION: First rewrite the function

$$\sqrt{x} = \sqrt{4 + (x-4)} = \sqrt{4\left(1 + \frac{x-4}{4}\right)} = 2\sqrt{1 + \frac{x-4}{4}}$$

Now find the MacLaurin series of $\sqrt{1+u}$ by setting $a = \frac{1}{2}$ in the binomial series formula.

$$(1+u)^{\frac{1}{2}} = \sqrt{1+u} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} u^n.$$

This is valid for $|u| < 1$. Now replace u by $\frac{x-4}{4}$ to get

$$\sqrt{1 + \frac{x-4}{4}} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{x-4}{4}\right)^n = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{4^n}(x-4)^n$$

This is valid for $\left|\frac{x-4}{4}\right| < 1$ or $|x-4| < 4$. So the radius of convergence is 4.

The final answer is:

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \frac{2}{4^n}(x-4)^n$$

If you're willing to do a lot of simplifying, you can eventually get to:

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n(2n-2)!}{2^{4n-2}(n!)^2}(x-4)^n$$