PRELIM 2 REVIEW QUESTIONS

Math 1910 Section 205/209

(1) Calculate the following integrals.

(a) $\int_0^1 \sqrt{1-x^2}\,dx$

SOLUTION: This is just the area under a semicircle of radius 1, so $\boxed{\pi/2}$.

(b) $\int \sin^2(x)\cos^4(x)\,dx$

SOLUTION:

$$\begin{aligned}\int \sin^2(x)\cos^4(x)\,dx &= \int (1-\cos^2(x))\cos^4(x)\,dx \\ &= \int \cos^4(x)\,dx - \int \cos^6(x)\,dx\end{aligned}$$

Now use the reduction formula.

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\int \cos^2(x)\,dx \\ &= \frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\int \frac{1}{2}(1+\cos(2x))\,dx \\ &= \frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{2}\sin(2x)\right) + C\end{aligned}$$

$$\begin{aligned}\int \cos^6(x)\,dx &= \frac{\sin(x)\cos^5(x)}{6} + \frac{5}{6}\int \cos^4(x) \\ &= \frac{\sin(x)\cos^5(x)}{6} + \frac{5}{6}\left(\frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{2}\sin(2x)\right) + C\right)\end{aligned}$$

Therefore, the answer is

$$\boxed{\frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{2}\sin(2x)\right) - \left(\frac{\sin(x)\cos^5(x)}{6} + \frac{5}{6}\left(\frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{2}\sin(2x)\right) + C\right)}$$

(c) $\int \sin^5(x)\cos^4(x)\,dx$

SOLUTION: This one looks like the reduction formula, but it's just substitution!

$$\begin{aligned}\int \sin^5(x)\cos^4(x)\,dx &= \int \sin(x)(1-\cos^2(x))^2\cos^4(x)\,dx \\ &= \int \sin(x)\cos^4(x)\left(1 - 2\cos^2(x) + \cos^4(x)\right)\,dx \\ &= \int \sin(x)\cos^4(x) - 2\sin(x)\cos^6(x) + \sin(x)\cos^8(x)\,dx\end{aligned}$$

Set $u = \cos(x)$, $du = -\sin(x)\,dx$. Therefore, we get

$$\begin{aligned}\int \sin^5(x)\cos^4(x)\,dx &= -\int u^4 - 2u^6 + u^8\,du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \\ &= \boxed{-\frac{\cos^5(x)}{5} + \frac{2\cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C}\end{aligned}$$

(d) $\int \tan^6(x)\sec^4(x)\,dx$

SOLUTION:

$$\int \tan^6(x)\sec^4(x)\,dx = \int \sec^2(x)\tan^6(x)(\tan^2(x)+1)\,dx$$

Let $u = \tan(x)$, $du = \sec^2(x)\,dx$.

$$\begin{aligned}\int \tan^6(x)\sec^4(x)\,dx &= \int u^6(u^2+1)\,du \\ &= \frac{u^9}{9} + \frac{u^7}{7} + C \\ &= \frac{\tan(x)^9}{9} + \frac{\tan(x)^7}{7} + C\end{aligned}$$

(e) $\int \cot^5(x)\csc^5(x)\,dx$

SOLUTION:

$$\int \cot^5(x)\csc^5(x)\,dx = \int \cot(x)\csc(x)(\csc^2(x)-1)^2\csc^4(x)\,dx$$

Let $u = \csc(x)$, $du = -\cot(x)\csc(x)\,dx$.

$$\begin{aligned}\int \cot^5(x)\csc^5(x)\,dx &= -\int (u^2-1)^2u^4\,du \\ &= -\int (u^4 - 2u^2 + 1)u^4\,du \\ &= -\int u^8 - 2u^6 + u^4\,du \\ &= \boxed{\frac{-\csc^9(x)}{9} + \frac{2\csc^7(x)}{7} - \frac{\csc^5(x)}{5} + C}\end{aligned}$$

(f) $\int \frac{x}{\sqrt{4-x^2}}\,dx$

SOLUTION: Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{4-x^2}}\,dx &= \int \frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}(2\cos\theta\,d\theta) \\
&= \int 2\sin\theta\,d\theta \\
&= -2\cos\theta + C \\
&= \boxed{-2\sqrt{4-x^2} + C}
\end{aligned}$$

In the last step, we have $x/2 = \sin\theta =$ opposite / hypotenuse, so $\cos\theta = \sqrt{4-x^2}$ (draw a triangle).

(g) $\displaystyle\int \frac{\cosh(x)}{\sinh^2(x)}\,dx$

SOLUTION: Let $u = \sinh(x)$ and $du = \cosh(x)\,dx$. Then

$$\int \frac{\cosh(x)}{\sinh^2(x)}\,dx = \int \frac{du}{u^2} = -u^{-1} + C = \boxed{-\frac{1}{\sinh(x)} + C.}$$

(h) $\displaystyle\int \sin^7(x)\cos^2(x)\,dx$

SOLUTION: Let $u = \cos(x)$, $du = -\sin(x)\,dx$. Then

$$\begin{aligned}
\int \sin^7 x\cos^2 x\,dx &= \int \sin(x)(1-\cos^2(x))^3\cos^2(x)\,dx \\
&= -\int (1-u^2)^3u^2\,du \\
&= -\int u^2 - 3u^4 + 3u^6 - u^8\,du \\
&= -\frac{u^3}{3} + \frac{3u^5}{5} - \frac{3u^7}{7} + \frac{u^9}{9} + C \\
&= \boxed{-\frac{\cos(x)^3}{3} + \frac{3\cos(x)^5}{5} - \frac{3\cos(x)^7}{7} + \frac{\cos(x)^9}{9} + C}
\end{aligned}$$

(i) $\displaystyle\int \frac{3x^2}{\sqrt{x^2-1}}\,dx$

SOLUTION: Let $x = \sec(\theta)$, $dx = \tan\theta\sec\theta\,d\theta$. Then

$$\begin{aligned}
\int \frac{3x^2}{\sqrt{x^2-1}}\,dx &= \int \frac{3\sec^2(\theta)}{\sqrt{\sec^2\theta - 1}}\tan\theta\sec\theta\,d\theta \\
&= \int 3\sec^3\theta\,d\theta
\end{aligned}$$

Now integrate by parts, with $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$ and $v = \tan\theta$, $dv = \sec^2\theta\,d\theta$.

$$
\begin{aligned}
3\int \sec^3\theta\,d\theta &= 3\left(\sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta\right) \\
\implies 3\sec^3\theta\,d\theta &= 3\left(\sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,d\theta\right) \\
\implies 3\int \sec^3(\theta)\,d\theta &= 3\sec\theta\tan\theta + 3\sec\theta\,d\theta - 3\int \sec^3\theta\,d\theta \\
\implies 6\int \sec^3(\theta)\,d\theta &= 3\sec\theta\tan\theta + 3\ln|\tan\theta + \sec\theta| + C \\
\implies 3\int \sec^3\theta\,d\theta &= \frac{\sec\theta}{2} + \frac{1}{2}\ln|\tan\theta + \sec\theta| + C.
\end{aligned}
$$

Therefore, the final answer is

$$
\boxed{\frac{3x}{2} + \frac{3}{2}\ln|\sqrt{x^2-1} + x| + C.}
$$

(j) $\displaystyle\int \frac{\cosh(x)}{3\sinh(x)+4}\,dx$

SOLUTION: Let $u = 3\sinh(x) + 4$, $du = 3\cosh(x)\,dx$. Then

$$
\int \frac{\cosh(x)}{3\sinh(x)+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln|3\sinh(x)+4| + C.}
$$

(k) $\displaystyle\int \frac{x^2+11x}{(x-1)(x+1)^2}\,dx$

SOLUTION: Perform partial fractions decomposition

$$
\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{(x+1)^2}
$$

Clear denominators first.

$$
x^2 + 11x = A(x+1)^2 + B(x-1)(x+1) + (Cx+D)(x-1)
$$

Plug in $x = 1$ to get $A = 3$. Plug in $x = -1$ to get $5 = D - C$. Plug in $x = 0$ to get $D = B - 3$. Plug in $x = 2$ to get $-1 = 3B + 2C + D$. Substitute the previous equations to get $B = 3$, $C = -5$, and $D = 0$. So

$$
\begin{aligned}
\int \frac{x^2+11x}{(x-1)(x+1)^2}\,dx &= \int \frac{3}{x-1} + \frac{3}{x+1} - \frac{5x}{(x+1)^2}\,dx \\
&= 3\ln|x-1| + 3\ln|x+1| - 5\int \frac{x}{(x+1)^2} \\
&= \boxed{3\ln|x-1| + 3\ln|x+1| - 5\ln|x+1| - \frac{5}{x+1} + C}
\end{aligned}
$$

(l) $\displaystyle\int \frac{3x^2-2}{x-4}\,dx$

SOLUTION: Do long division. Divide $3x^2 - 2$ by $x - 4$ to get

$$\int \frac{3x^2-2}{x-4}\,dx = \int (3x+12) + \frac{46}{x-4}\,dx = \boxed{\frac{3x^2}{2} + 12x + 46\ln|x-4| + C.}$$

(m) $\int \coth^2(1-4t)\,dt$

SOLUTION: Let $u = 1-4t$, $du = -4\,dt$. Then

$$\begin{aligned}\int \coth^2(1-4t)\,dt &= \frac{-1}{4}\int \coth^2(u)\,du \\ &= -\frac{1}{4}\int \operatorname{csch}^2(u) + 1\,du \\ &= -\frac{1}{4}(-\coth(u) + u + C) \\ &= \boxed{\frac{1}{4}\coth(1-4t) - (1-4t) + C.}\end{aligned}$$

(n) $\int \frac{1}{x^2+4x-5}\,dx$

SOLUTION: Perform partial fractions decomposition:

$$\frac{1}{x^2+4x-5} = \frac{A}{x+5} + \frac{B}{x+1} \implies 1 = A(x-1) + B(x+5)$$

Set $x = 1 \implies B = 1/6$. Set $x = -5 \implies A = -1/6$. Then

$$\int \frac{1}{x^2+4x-5} = -\frac{1}{6}\int \frac{1}{x+5} - \frac{1}{x-1}\,dx = \boxed{-\frac{1}{6}\left(\ln|x+5| - \ln|x-1| + C\right)}$$

(2) Find the volume of the solid obtained by rotating $y = x\sqrt{1-x^2}$ about the y-axis.

SOLUTION: Using the cylindrical shells method:

$$\begin{aligned}V &= \int_0^1 2\pi x\left(x\sqrt{1-x^2}\right)\,dx \\ &= 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx\end{aligned}$$

Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$. Then

$$V = 2\pi\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta$$

Let $u = \sin\theta$, $du = \cos\theta\,d\theta$. Then

$$V = 2\pi\int_0^1 u^2\,du = 2\pi\frac{u^3}{3}\Big|_0^1 = \boxed{\frac{2}{3}\pi.}$$

(3) Find the arc length of the graph of $y = \tan(x)$ over the interval $[0, {}^{\pi}/_{4}]$.

SOLUTION: Plug in to the arc length formula.

$$\int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \sec^4(x)}\, dx.$$

No antiderivative exists, so numerical integration techniques must be used.

(4) Suppose that a random variable X is distributed with density $p(x) = C\sqrt{1 - x^2}$ on $[-1, 1]$. Find C such that $p(x)$ defines a probability density function, and compute $P(-{}^{1}/_{2} \leq X \leq 1)$.

SOLUTION: To find C, set

$$1 = \int_{-1}^{1} C\sqrt{1 - x^2}\, dx$$

Then evaluate the integral on the right and solve for C. To evaluate the integral, let $x = \sin\theta$, $dx = \cos\theta\, d\theta$. Then

$$\begin{aligned}
\int_{-1}^{1} C\sqrt{1 - x^2}\, dx &= C\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\
&= C\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos(2\theta))\, d\theta \qquad = \frac{C}{2}\int_{-\pi/2}^{\pi/2} 1\, d\theta + \frac{C}{2}\int_{-\pi/2}^{\pi/2} \cos(2\theta)\, d\theta \\
&= C\frac{\pi}{2} + \frac{C}{2}\left(\frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= C\frac{\pi}{2} + \frac{C}{4}\left(\sin\pi - \sin(-\pi)\right) \\
&= C\frac{\pi}{2}.
\end{aligned}$$

Therefore, $C = 2/\pi$.

Now to find the probability, we've already computed the antiderivative of $p(x)$, so we just need to substitute new bounds. The probability is

$$P(-1/2 \leq X \leq 1) = \int_{-1/2}^{1} \frac{2}{\pi}\sqrt{1 - x^2}\, dx = \frac{2}{\pi}\left(\sin^{-1}(x) + \frac{x\sqrt{1 - x^2}}{2}\right)\Bigg|_{-1/2}^{1} = \boxed{\frac{2}{3} + \frac{\sqrt{3}}{4\pi}}$$

(5) Find C such that $p(x) = Ce^{-x}e^{-e^{-x}}$ is a probability density function on $(-\infty, \infty)$.

SOLUTION: This is one of the homework questions from last week (also it was on the quiz). Go look at the homework solutions on blackboard. The answer is $\boxed{C = 1}$.

(6) Suppose that a random variable X is distributed with density $p(x) = x^2 e^{-x^2}$ on $(-\infty, \infty)$. Find the mean of X.

SOLUTION:

$$\mu = \int_{-\infty}^{\infty} x p(x)\, dx = \int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx$$

This is an odd function integrated over a symmetric domain, so the integral is $\boxed{\text{zero}}$.

(7) Suppose that a random variable X is distributed with density $p(x) = \frac{1}{r}e^{-x/r}$ on $(0, \infty)$. Find the mean of X.

SOLUTION: This was on the homework last week, and I also did it in class. Go see the homework solutions on blackboard. The answer is $\boxed{\mu = r}$.

(8) Calculate T_6 for the integral $I = \displaystyle\int_0^2 x^3\,dx$.

SOLUTION: $\Delta x = (2-0)/6 = \frac{1}{3}$. This is tedious, but easy to do. The answer is

$$\boxed{T_6 = \frac{111}{27}}$$

(a) Is T_6 too large or too small? Explain graphically.
SOLUTION: Between $x = 0$ and $x = 2$, the graph of $y = x^3$ is concave up, so trapezoid rule overestimates the area under the graph; the trapezoids are above the graph.

(b) Show that $K_2 = |f''(2)|$ may be used in the error bound and find a bound for the error.
SOLUTION: K_2 is the max value of $|f''(x)|$ on the interval $[0, 2]$. $f(x) = x^3$, so $f''(x) = 6x$. Therefore, the maximum value of $|f''(x)|$ on the interval $[0, 2]$ happens at $x = 2$, and $K_2 = |f''(2)| = 12$. Finally,

$$\text{Error} \le \frac{K_2(b-a)}{6n^2} = \frac{12(2-0)}{6(6^2)} = \frac{24}{216} = \frac{1}{9}.$$

(c) Evaluate I and check that the actual error is less than the bound computed in (b).
SOLUTION: This is easy to integrate.

$$\int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = 4$$

So the actual error is

$$\text{Error} = \left|4 - \frac{111}{27}\right| = \frac{1}{9}.$$

This is less than the error bound, which says that the error is at most 1/9.

(9) Radium-226 has a half-life of 1590 years. Consider a mass of 100 mg of Radium-226.

(a) What is the mass of Radium remaining after 1000 years?
SOLUTION: The equation for radioactive decay is exponential decay,

$$m(t) = m_0 e^{-t/T}$$

where m_0 is the inital mass, $m_0 = 100$ mg, $m(t)$ is the number of milligrams remaining after t years, and T is the half life, $T = 15(0$ years. Then

$$m(1000) = 100e^{-1000/1590}\boxed{\approx 187.5}$$

(b) When will the mass of Radium be 10 mg?

SOLUTION: We want to know for which t does $m(t) = 10$ mg. So set

$$10 = 100e^{-t/1590} \implies \frac{1}{10} = -\frac{t}{1590} \implies \boxed{t = -1590\ln(1/10)}.$$

(10) Show that $\int_1^\infty e^{-x^2}\,dx$ converges using the Comparison Theorem.

SOLUTION: The comparison theorem says that

$$f(x) \le g(x) \implies \int_1^\infty f(x)\,dx \le \int_1^\infty g(x)\,dx.$$

On the interval $[1, \infty)$, $e^{-x^2} \le e^{-x}$. Therefore,

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

Hence, the integral converges.