DERIVATIVES REVIEW

Math 1910

NAME: SOLUTIONS

August 24, 2017

RAPID REVIEW

(1) Given a function f, the **derivative** of f at the point a is defined by

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(2) The line tangent to $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$.

(3) Differentiation rules:

(a) $(cf)' = cf'$ if c is a constant.

(b) $(f+g)' = f' + g'$

(c) **Product rule**: $(fg)' = f'g + fg'$.

(d) **Quotient rule**: $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$.

(e) **Chain rule**: $(f(g(x)))' = f'(g(x))g'(x)$.

(4) **Implicit differentiation** is used to compute $\frac{dy}{dx}$ when the variables x and y are related by an equation, such as $x^3 - y^3 = 4$. This is a special instance of the chain rule. To perform implicit differentiation, take the derivative of both sides. Remember that y is a function of x, so $\frac{d}{dx} f(y) = f'(y)y'$.

(5) **The first derivative test:** If f is differentiable and c is a critical point, then the type of critical point can be found in the table.

Sign Change of $f'(x)$	**Type of Critical Point**
From + to −	Local max
From − to +	Local min

(6) A function f is **concave up** on (a, b) if f' is increasing, and **concave down** if f is decreasing. A **point of inflection** is a point $(c, f(c))$ where the concavity changes. We can use the first derivative test on the derivative f' to find the inflection points of f.

PROBLEMS

(1) Compute $\frac{dy}{dx}$.

(a) $y = 3x^5 - 7x^2 + 4$

SOLUTION: $15x^4 - 14x$

(b) $y = \frac{x}{x^2+1}$

SOLUTION: $\frac{1-x^2}{(x^2+1)^2}$

(c) $y = (x^4 - 9x)^6$

SOLUTION: $6\left(x^4 - 9x\right)^5 (4x^3 - 9)$

(d) $y = \sqrt{x + \sqrt{x}}$

SOLUTION: $\dfrac{\frac{1}{2\sqrt{x}} + 1}{2\sqrt{x + \sqrt{x}}}$

(e) $y = \tan(x)$

SOLUTION: $\sec^2(x)$

(f) $y = \sin(2x)\cos^2(x)$

SOLUTION: $2\cos^2(x)(2\cos(2x) - 1)$

(g) $y = \tan(\sqrt{1 + \csc x})$

SOLUTION: $-\dfrac{\cot(x)\csc(x)\sec^2\left(\sqrt{\csc(x)+1}\right)}{2\sqrt{\csc(x)+1}}$

(h) $x^3 - y^3 = 4$

SOLUTION: Use implicit differentiation. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ when $y \neq 0$.

(i) $y = xy^2 + 2x^2$

SOLUTION: Use implicit differentiation. $\dfrac{dy}{dx} = \dfrac{4x + y^2}{1 - 2xy}$

(j) $y = \sin(x + y)$

SOLUTION: Use implicit differentiation. $\dfrac{dy}{dx} = \dfrac{\cos(x + y)}{1 - \cos(x + y)}$

(2) Find the points on the graph of $f(x) = x^3 - 3x^2 + x + 4$ where the tangent line has slope 10.

SOLUTION: The points are $(-1, -1)$, $(3, 7)$.

(3) Find the critical points of f and determine if they are minima or maxima.

(a) $f(x) = x^3 - 4x^2 + 4x$

SOLUTION: maximum at $x = \frac{2}{3}$ and minimum at $x = 2$

(b) $f(x) = x^2(x + 2)^3$

SOLUTION: maximum at $x = \frac{-4}{5}$; minimum at $x = 0$

(c) $f(x) = x^{2/3}(1 - x)$

SOLUTION: maximum at $x = \frac{2}{5}$; minimum at $x = 0$

(4) Find the points of inflection of the function f

(a) $f(x) = x^3 - 4x^2 + 4x$

SOLUTION: at $x = \frac{4}{3}$

(b) $f(x) = x - 2\cos x$

SOLUTION: at $x = \frac{(2n+1)\pi}{2}$ for all integers n

(c) $f(x) = \frac{x^2}{x^2+4}$

SOLUTION: at $x = \pm\frac{2}{\sqrt{3}}$

(5) Find conditions on a and b that ensure $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$.

SOLUTION: Whenever $a \geq 0$.