§5.1 Approximating Areas
§5.2 Definite Integrals

Math 1910

Name: Solutions

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Rapid Review

(1) Approximations to the area under the graph of f over the interval $[a, b]$:

Right-endpoint	Left-endpoint	Midpoint
$R_N = \Delta x \sum_{j=1}^{N} f(x_j)$ (1), (2)	$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$ (3), (4)	$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$ (5)

(2) If f is continuous on $[a, b]$, then the area A under the graph $y = f(x)$ is defined as

$$A := \lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_n \quad (6)$$

(3) The **definite integral** is the signed area (7) of the region between the graph of f and the x-axis. If f is continuous (8) on $[a, b]$, then f is integrable over $[a, b]$.

(4) Some properties of definite integrals:

(a) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ (9)

(b) $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$ (10)

(c) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(d) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ (11)

(5) Some formulas for computing integrals

(a) $\int_a^b C dx = C(b - a)$ (12)

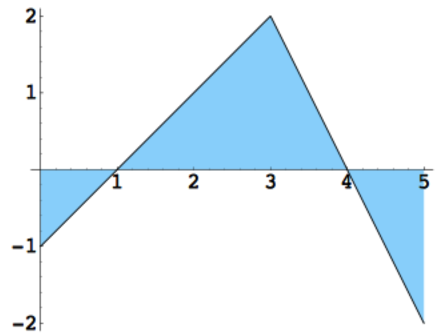
(b) $\int_0^b x dx = \frac{1}{2}b^2$ (13)

(c) $\int_0^b x^2 dx = \frac{1}{3}b^3$ (14)

(6) **Comparison Theorem:** If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \le$ (15) $\int_a^b g(x)\,dx$.

PROBLEMS

(1) Use the graph of $g(x)$ given below to evaluate the following integrals.



(a) $\int_0^3 g(x)\,dx$

SOLUTION: The region bounded by the curve $y = g(x)$ and the x-axis over the interval $[0, 3]$ is comprised of two right triangles, one with area $\frac{1}{2}$ below the axis, and one with area 2 above the axis. The definite integral is therefore equal to $2 - \frac{1}{2} = \frac{3}{2}$.

(b) $\int_3^5 g(x)\,dx$

SOLUTION: The region bounded by the curve $y = g(x)$ and the x-axis over the interval $[3, 5]$ is comprised of another two right triangles, one with area 1 above the axis and one with area 1 below the axis. The definite integral is therefore equal to zero.

(c) $\int_0^5 g(x)\,dx$

SOLUTION: This is the sum of the previous two integrals, by our integral properties.

$$\int_0^5 g(x)\,dx = \int_0^3 g(x)\,dx + \int_3^5 g(x)\,dx = \frac{3}{2} + 0 = \frac{3}{2}$$

(2) Find a formula for R_N for $f(x) = 3x^2 - x + 4$ over the interval $[0, 1]$.

SOLUTION: We have $\Delta x = \frac{1-0}{N} = \frac{1}{N}$. We will use the right endpoint, so $x_j = 0 + j\Delta x$. So using the formula for R_N, we have

$$\begin{aligned} R_N &= \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) \\ &= \frac{1}{N}\sum_{j=1}^{N} f\left(\frac{j}{N}\right) \\ &= \frac{1}{N}\sum_{j=1}^{N}\left(3\frac{j^2}{N^2} - \frac{j}{N} + 4\right) \\ &= \frac{3}{N^3}\sum_{j=1}^{N} j^2 - \frac{1}{N^2}\sum_{j=1}^{N} j + \frac{4}{N}\sum_{j=1}^{N} 1 \\ &= \frac{3}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) - \frac{1}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{4}{N}N \\ &= 1 + \frac{3}{2N} + \frac{1}{2N^2} - \frac{1}{2} - \frac{1}{2N} + 4 \end{aligned}$$

(3) Calculate $\int_2^5 (2x+1)\,dx$ in three ways:

(a) As a limit $\lim_{N\to\infty} R_N$.

SOLUTION:

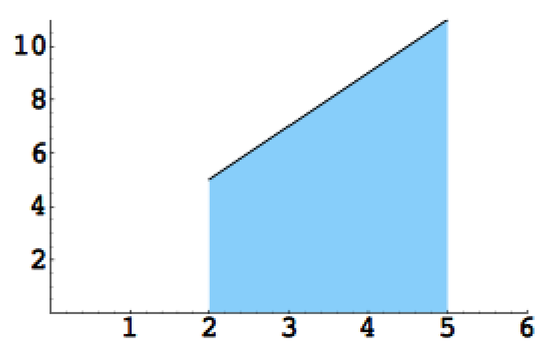
$$R_N = \sum_{k=1}^{N}\left(2\left(2 + \frac{3k}{N}\right) + 1\right)\frac{3}{N} = \sum_{k=1}^{N}\left(\frac{15}{N} + \frac{18k}{N^2}\right) = 15 + \frac{18}{N^2}\frac{N(N+1)}{2} = 15 + 18\left(\frac{1}{2} + \frac{1}{2N}\right)$$

Then taking the limit as $N \to \infty$, we see that

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(15 + 9 + \frac{1}{2N}\right) = 24.$$

(b) Using geometry, interpreting this as the area under a graph.

SOLUTION: This is the area of the trapezoid pictured below.



Which is $3((11+5)/2) = 24$.

(c) Using the properties of the integral.

SOLUTION:

$$\begin{aligned}
\int_2^5 (2x+1)\,dx &= \int_2^5 2x\,dx + \int_2^5 1\,dx \\
&= 2\int_2^5 x\,dx + (5-2)(1) \\
&= 2\left(\int_0^5 x\,dx + \int_2^0 x\,dx\right) + 3 \\
&= 2\left(\frac{1}{2}5^2 - \int_0^2 x\,dx\right) + 3 \\
&= 2\left(\frac{25}{2} - \frac{1}{2}2^2\right) + 3 \\
&= 2\left(\frac{21}{2}\right) + 3 = 21 + 3 = 24
\end{aligned}$$

(4) Use the basic properties of the integral to calculate the following.

(a) $\int_1^4 6x^2\,dx$

SOLUTION:

$$\int_1^4 6x^2 dx = 6\int_0^4 x^2 dx - 6\int_0^1 x^2\,dx = 6\left(\frac{1}{3}(4)^3 - \frac{1}{3}(1)^3\right) = 126.$$

(b) $\int_{-2}^3 (3x+4)\,dx$

SOLUTION:

$$\begin{aligned}
\int_{-2}^3 (3x+4)dx &= 3\int_{-2}^3 x\,dx + 4\int_{-2}^3 dx \\
&= 3\left(\int_{-2}^0 x dx + \int_0^3 x\,dx\right) + 4(3-(-2)) \\
&= 3\left(\int_0^3 x\,dx - \int_0^{-2} x\,dx\right) + 20 \\
&= 3\left(\frac{1}{2}3^2 - \frac{1}{2}(-2)^2\right) + 20 = \frac{55}{2}
\end{aligned}$$

(c) $\int_1^3 |2x-4|\,dx$

SOLUTION: The area between $|2x-4|$ and the x axis consists of two triangles above the x-axis, each with width 1 and height 2, and hence with area 1. The total area, and hence the definite integral, is 2.

(5) Evaluate $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \sqrt{1-\left(\frac{j}{N}\right)^2}$ by interpreting the limit as an area.

SOLUTION: The limit represents the area between the graph of $y = f(x) = \sqrt{1-x^2}$ and the x-axis over the interval $[0,1]$. This is the portion of the circular disk $x^2 + y^2 \le 1$ that lies in the first quadrant. Accordingly, its area is $\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$.