§5.3: Indefinite Integrals
§5.4, §5.5: Fundamental Theorem

Name: Solutions

Math 1910

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One-page Review

(1) F is called an **antiderivative** of f if $F'(x) = f(x)$ (1). Any two antiderivatives of f on an interval (a, b) differ by a constant.

(2) **Fundamental Theorem of Calculus, Part I (FTC I):** if $F(x)$ is an antiderivative for $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad (2)$$

(3) (a) $\int 0\,dx = C$ (3)

(b) $\int k\,dx = kx + C$ (4)

(c) $\int cf(x)\,dx = c\int f(x)\,dx$ (5)

(d) $\int (f(x) + g(x))\,dx = \int f(x)\,dx$ (6) $+ \int g(x)\,dx$ (7)

(e) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (8)

(f) $\int \sin x\,dx = -\cos x + C$ (9)

(g) $\int \sec^2 x\,dx = \tan x + C$ (10)

(h) $\int \sec x \tan x\,dx = \sec x + C$ (11)

(4) To solve an initial value problem $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$, first find the general antiderivative $y = F(x) + C$. Then determine C using the initial condition $F(x_0) + C = y_0$.

(5) The **area function** with lower limit a is $A(x) = \int_a^x f(t)\,dt$ (12).

(6) **Fundamental Theorem of Calculus, Part II (FTC II):**

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad (13)$$

(7) A consequence of FTC II is that *every* continuous function has an antiderivative.

(8) Let $G(x) = \int_a^{g(x)} f(t)\,dt$. Let $A(x) = \int_a^x f(t)\,dt$. Then

$$\frac{d}{dx}G(x) = \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}A(g(x)) = A'(g(x))g'(x) = f(g(x))g'(x) \quad (14)$$

PROBLEMS

(1) Evaluate the integral:

(a) $\int \cos x \, dx$

SOLUTION: $\int \cos x \, dx = \sin x + C$

(b) $\int \csc x \cot x \, dx$

SOLUTION: $\int \csc x \cot x \, dx = -\csc x + C$

(c) $\int \frac{3}{x^{3/2}} \, dx$

SOLUTION: Since $\frac{3}{x^{3/2}} \, dx = 3x^{-3/2}$, we get

$$\begin{aligned}\int \frac{3}{x^{3/2}} \, dx &= \int 3x^{-3/2} \, dx \\ &= 3\left(\frac{-1}{(-1/2)} x^{-1/2}\right) + C \\ &= -6x^{-1/2} + C\end{aligned}$$

(d) $\int_{-2}^{2} (10x^9 + 3x^5) \, dx$

SOLUTION: $\int_{-2}^{2} (10x^9 + 3x^5) \, dx = \left(x^{10} + \frac{1}{2}x^6\right)\Big|_{-2}^{2} = \left(2^{10} + \frac{1}{2}2^6\right) - \left(2^{10} + \frac{1}{2}2^6\right) = 0$

(e) $\int_0^4 \sqrt{x} \, dx$

SOLUTION: $\int_0^4 \sqrt{x} \, dx = \int_0^4 x^{1/2} \, dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$

(f) $\int_{\pi/4}^{3\pi/4} \sin\theta \, d\theta$

SOLUTION: $\int_{\pi/4}^{3\pi/4} \sin\theta \, d\theta = -\cos\theta\Big|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

(g) $\int_0^5 |x^2 - 4x + 3| \, dx$

SOLUTION: Write the integral as a sum of integrals without absolute values and then apply FTC I.

$$\begin{aligned}\int_9^5 |x^2 - 4x + 3| \, dx &= \int_0^5 |(x-3)(x-1)| \, dx \\ &= \int_0^1 (x^2 - 4x + 3) \, dx + \int_1^3 (-x^2 - 4x + 3) \, dx + \int_3^5 (x^2 - 4x + 3) \, dx \\ &= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_1^2 + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_3^5 \\ &= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{125}{3} - 50 + 15\right) - (9 - 18 + 9) \\ &= \frac{28}{3}\end{aligned}$$

(h) $\displaystyle\int_4^9 \frac{16+t}{t^2}\,dt$

SOLUTION: $\displaystyle\int_4^9 \frac{16+t}{t^2}\,dt = \int_4^9 16t^2 + t^{-1}\,dt = -16t^{-1} + \log t\Big|_4^9 = \frac{20}{9} + \log\frac{9}{4}$

(2) Solve the differential equation $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$ with initial condition $y(1) = 1$.

SOLUTION: Since $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$, then

$$y = \int (8x^3 + 3x^2 - 3)\,dx = 2x^4 + x^3 - 3x + C$$

Thus $1 = y(1) = 0 + C$, and so $C = 1$. Therefore, $y = 2x^4 + x^3 - 3x + 1$.

(3) Given that $f''(x) = x^3 - 2x + 1$, $f'(0) = 1$, and $f(0) = 0$, find f' and then find f.

SOLUTION: Let $g(x) = f'(x)$. The statement gives that $g'(x) = x^3 - 2x + 1$, $g(0) = 1$. From this initial value problem, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$. Then $g(0) = 1$ gives $C = 1$, so $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1$.

Now we have a new initial value problem to find f, namely $f'(x) = \frac{1}{4}x^4 - x^2 + x + 1$ and $f(0) = 0$. So we get that $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$. Then $f(0) = 0$ gives $C = 0$, so

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x.$$

(4) If $G(x) = \int_1^x \tan t\,dt$, find $G(1)$ and $G'(\pi/4)$.

SOLUTION: By definition, $G(1) = \displaystyle\int_1^1 \tan t\,dt = 0$. By FTC II, $G'(x) = \tan x$, so $G'(\pi/4) = \tan(\pi/4) = 1$.

(5) Find a formula for the function represented by the integral: $\int_2^x (t^2 - t)\,dt$.

SOLUTION: $\displaystyle\int_2^x (t^2 - t)\,dt = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right)\Big|_2^\pi = \frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{2}{3}$

(6) Express the antiderivative $F(x)$ of $f(x)$ as an integral, given that $f(x) = \sqrt{x^4+1}$ and $F(3) = 0$.

SOLUTION: The antiderivative $F(x)$ of $f(x) = \sqrt{x^4+1}$ satisfying $F(3) = 0$ is

$$F(x) = \int_3^x \sqrt{t^4+1}\,dt$$

(7) Calculate the derivative: $\displaystyle\frac{d}{dx}\int_1^{x^3} \tan t\,dt$.

SOLUTION: By combining FTC II and the chain rule. Let $G(x) = \displaystyle\int_1^{x^3} \tan t\,dt$, $A(x) = \displaystyle\int_1^x \tan t\,dt$, $g(x) = x^3$. Then $G(x) = A(g(x))$, so we can use the chain rule.

$$G'(x) = A'(g(x))g'(x) = \tan x^3(3x^2) = 3x^2\tan x^3$$