§5.7: Substitution

Math 1910

Name: Solutions

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One Page Review

- Try the **Substitution Method** when the integrand has the form $f(u(x))u'(x)$. If F is an antiderivative of f, then

$$\int f(u(x))u'(x)\,dx = \boxed{F(u(x))}^{(1)} + C$$

- The differential of $u(x)$ is related to dx by $du = \boxed{u'(x)\,dx}^{(2)}$.
- The **Change of Variables Formula** says that
  - For indefinite integrals: $\int f(u(x))u'(x)\,dx = \boxed{\int f(u)\,du}^{(3)}$
  - For definite integrals: $\int_a^b f(u(x))u'(x)\,dx = \boxed{\int_{u(a)}^{u(b)} f(u)\,du}^{(4)}$

PROBLEMS

(1) Evaluate the indefinite integral.

(a) $\int x(x+1)^9\,dx$

SOLUTION: Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence,

$$\int x(x+1)^9\,dx = \int (u-1)u^9\,du = \int (u^{10} - u^9)\,du$$
$$= \frac{1}{11}u^{11} - \frac{1}{10}u^{10} + C = \frac{1}{11}(x+1)^{11} - \frac{1}{10}(x+1)^{10} + C$$

(b) $\int \sin(2x-4)\,dx$

SOLUTION: Let $u = 2x-4$. Then $du = 2dx \implies \frac{1}{2}du = dx$. So

$$\int \sin(2x-4)\,dx = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(2x-4) + C$$

(c) $\int \frac{x^3}{(x^4+1)^4}\,dx$

SOLUTION: Let $u = x^4+1$. Then $du = 4x^3\,dx$ or $\frac{1}{4}\,du = x^3\,dx$. Hence

$$\int \frac{x^3}{(x^4+1)^4}\,dx = \frac{1}{4}\int \frac{1}{u^4}\,du = -\frac{1}{12}u^{-3} + C = -\frac{1}{12}(x^4+1)^{-3} + C$$

(d) $\int \sqrt{4x-1}\,dx$

SOLUTION: Let $u = 4x-1$. Then $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Hence,

$$\int \sqrt{4x-1}\,dx = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{6}(4x-1)^{3/2} + C$$

(e) $\int x\cos(x^2)\,dx$

SOLUTION: Let $u = x^2$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence,

$$\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

(f) $\int \sin^5 x\cos x\,dx$

SOLUTION: Let $u = \sin x$. Then $du = \cos x\,dx$. Hence,

$$\int \sin^5 x\cos x\,dx = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C.$$

(g) $\int \sec^2 x\tan^4 x\,dx$

SOLUTION: Let $u = \tan x$. Then $du = \sec^2 x\,dx$. Hence,

$$\int \sec^2 x\tan^4 x\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C.$$

(h) $\displaystyle\int \frac{dx}{(2+\sqrt{x})^3}$

SOLUTION: Let $u = 2+\sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so that

$$2\sqrt{x}\,du = dx \implies 2(u-2)\,du = dx.$$

Using this, we get

$$\begin{aligned}
\int \frac{dx}{(2+\sqrt{x})^3} &= \int 2\frac{u-2}{u^3}\,du \\
&= 2\int (u^{-2}-2u^{-3})\,du \\
&= 2(-u^{-1}+u^{-2})+C \\
&= 2\left(-\frac{1}{2+\sqrt{x}}+\frac{1}{(2+\sqrt{x})^2}\right)+C \\
&= 2\left(\frac{-2-\sqrt{x}+1}{(2+\sqrt{x})^2}\right)+C \\
&= -2\frac{1+\sqrt{x}}{(2+\sqrt{x})^2}+C
\end{aligned}$$

(2) Evaluate the definite integral.

(a) $\displaystyle\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION: Let $u = x^2+1$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$. Hence,

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \frac{1}{2}\cdot -\frac{1}{2}u^{-2}\Big|_1^2 = -\frac{1}{16}+\frac{1}{4} = \frac{3}{16}$$

(b) $\displaystyle\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION: Let $u = x-9$. Then $du = dx$. Hence,

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\Big|_1^8 = 3(2-1) = 3$$

(c) $\displaystyle\int_{-8}^{8} \frac{x^{15}}{3+\cos^2 x}\,dx$

SOLUTION: This function is odd! Set $f(x) = \frac{x^{15}}{3+\cos^2 x}$, and then $f(-x) = -f(x)$. The bounds of the integral are symmetric, and the function is odd, so the answer is zero.

(d) $\displaystyle\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta$

SOLUTION: Let $u = \cos\theta$; then $du = -\sin\theta\,d\theta$, and the new bounds of integration are $\cos 0 = 1$ to $\cos\pi/2 = 0$. Thus,

$$\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta = -\int_1^0 \sec^2 u\,du = \tan u\Big|_0^1 = \tan 1.$$

(e) $\displaystyle\int_{-4}^{-2} \frac{12x\,dx}{(x^2+2)^3}$

SOLUTION: Let $u = x^2+2$; then $du = 2x\,dx$ and the new bounds of integration are $u = 18$ to $u = 6$. Thus,

$$\int_{-4}^{-2} \frac{12x\,dx}{(x^2+2)^3} = \int_{18}^{6} \frac{6}{u^3}\,du = -3u^2\Big|_{18}^{6} = -\frac{2}{27}$$

(f) $\displaystyle\int_{1}^{8} \sqrt{t+8}\,dt$

SOLUTION: Let $u = t+8$; then $t^2 = (u-8)^2$ and $du = dt$. The new bounds of integration are $u = 9$ to $u = 16$. Thus,

$$\begin{aligned}\int_{1}^{8} t^2\sqrt{t+8}\,dt = \int_{9}^{16} (u-8)^2\sqrt{u}\,du &= \int_{9}^{16} \left(u^{5/2} - 16u^{3/2} + 64u^{1/2}\right)\,du \\ &= \left(\frac{2}{7}u^{7/2} - \frac{32}{5}u^{5/2} + \frac{128}{3}u^{3/2}\right)\Big|_{9}^{16} = \frac{66868}{105}\end{aligned}$$

(g) $\displaystyle\int_{0}^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta$

SOLUTION: Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and when $\theta = 0$, $u = 1$ and when $\theta = \pi/3$, $u = \frac{1}{7}2$. So

$$\int_{0}^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta = -\int_{1}^{1/2} u^{-2/3}\,du = -3u^{1/3}\Big|_{1}^{1/2} = -3(2^{-1/3}-1) = 3 - \frac{3\sqrt[3]{4}}{2}.$$

(h) $\displaystyle\int_{-2}^{4} |(x-1)(x-3)|\,dx$

SOLUTION:

$$\begin{aligned}\int_{-2}^{4} |(x-1)(x-3)|\,dx &= \int_{-2}^{1} (x^2-4x+3)\,dx + \int_{1}^{3} (-x^2+4x-3)\,dx + \int_{3}^{4} (x^2-4x+3)\,dx \\ &= \left(\frac{1}{3}x^3 - 2x62 + 3x\right)\Big|_{-2}^{1} + \left(-\frac{1}{3}x^3 + 2x^2 - 3x\right)\Big|_{1}^{3} + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_{3}^{4} \\ &= \frac{4}{3} - \left(-\frac{50}{3}\right) + 0 - \left(-\frac{4}{3}\right) + \frac{4}{3} - 0 \\ &= \frac{62}{3}\end{aligned}$$

(3) Evaluate the indefinite integral

$$\int \tan x \sec^2 x\,dx$$

in two ways: first using $u = \tan x$ and then using $u = \sec x$. What's going on here?

SOLUTION: The two substitutions yield two different antiderivatives: $\frac{1}{2}\tan^2 x + C$ and $\frac{1}{2}\sec^2 x + C$. But recall that two antiderivatives for a function must differ by a constant! Indeed, using the identity $\tan^2 x + 1 = \sec^2 x$, we see that

$$\frac{1}{2}\sec^2 x - \frac{1}{2}\tan^2 x = \frac{1}{2}.$$