§6.1: Areas, §6.2: Volumes, §6.3: Revolution

Math 1910

Name: Solutions

September 19, 2017

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One Page Review

(1) The graph of $x = f(y)$ is the graph of $y = f(x)$ reflected across the line $y = x$ (1).

(2) The area between $f(x)$ and $g(x)$ from a to b is $\int_a^b (y_{top} - y_{bottom})\, dx$ (2).

(3) The average value (3) of $f(x)$ over the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\, dx$ (4).

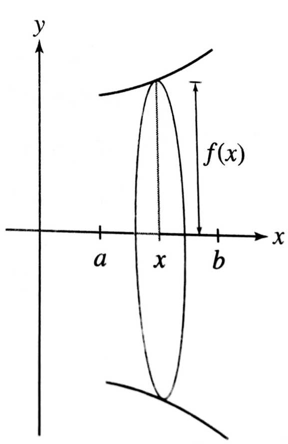
(4) The **Mean Value Theorem for Integrals** says that if f is continuous on $[a, b]$ with mean value M, then there is some $c \in [a, b]$ such that $f(c) = M$ (5).

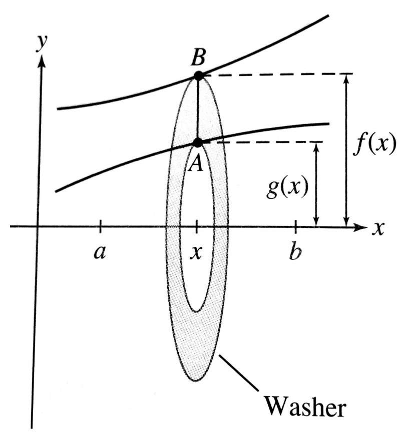
(5) If a shape has cross-sectional area $A(y)$ and height extends from $y = a$ to $y = b$, then it's volume is $\int_a^b A(y)\, dy$ (6).

(6) **Cavilieri's Principle** says if two solids have equal cross-sectional areas (7), then they also have equal volumes (8).

(7) **The Disk Method:** If $f(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region under the graph around the x-axis has volume $\int_a^b \pi f(x)^2\, dx$ (9).

(8) **The Washer Method:** If $f(x) \geq g(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region between $f(x)$ and $g(x)$ around the x-axis has volume $\int_a^b \pi(f(x)^2 - g(x)^2)\, dx$ (10).





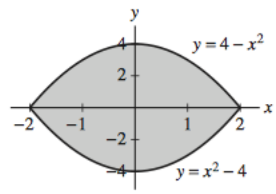
PROBLEMS

(1) Sketch the region enclosed by the curves and set up an integral to compute it's area, but do not evaluate.

(a) $y = 4 - x^2, y = x^2 - 4$

SOLUTION: Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4)\right) dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left(8x - \frac{2}{3}x^3\right)\Big|_{-2}^{2} = \frac{64}{3}.$$



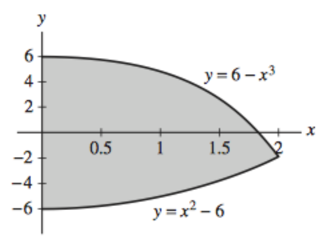
(b) $y = x^2 - 6, y = 6 - x^3, x = 0$

SOLUTION: Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval $[0, 2]$; hence, the area of the region enclosed by these curves and the y-axis is

$$\int_{0}^{2} \left((6 - x^3) - (x^2 - 6)\right) dx = \int_{0}^{2} (-x^3 - x^2 + 12)\, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x\right)\Big|_{0}^{2} = \frac{52}{3}$$



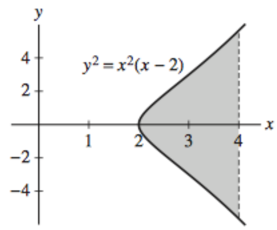
(c) $y = x\sqrt{x - 2}, y = -x\sqrt{x - 2}, x = 4$

SOLUTION: Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x - 2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_{2}^{4} \left(x\sqrt{x-2} - (-x\sqrt{x-2})\right) dx = \int_{2}^{4} 2x\sqrt{x-2}\, dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

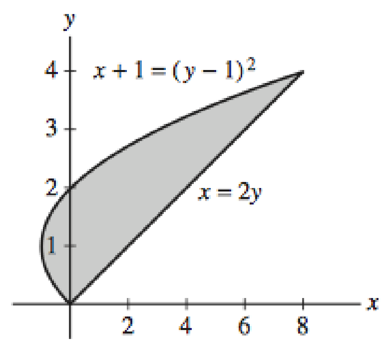
$$\int_{2}^{4} 2x\sqrt{x-2}\, dx = \int_{0}^{2} 2(u+2)\sqrt{u}\, du = \int_{0}^{2} \left(2u^{3/2} + 4u^{1/2}\right) du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right)\Big|_{0}^{2} = \frac{128\sqrt{2}}{15}.$$



(d) $x = 2y, x + 1 = (y-1)^2$

SOLUTION: Setting $2y = (y-1)^2 - 1$ yields $0 = y^2 - 4y = y(y-4)$, so the two curves intersect at $y = 0$ and $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

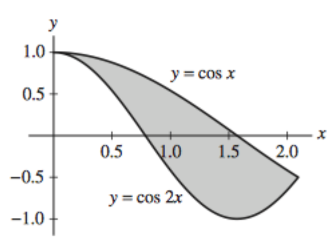
$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right)\, dy = \int_0^4 (4y - y^2)\, dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



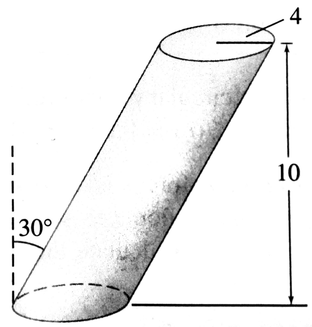
(e) $y = \cos x, y = \cos(2x), x = 0, x = \frac{2\pi}{3}$

SOLUTION: From the graph below, we see that $y = \cos x$ lies above $y = \cos 2x$ over the interval $[0, \frac{2\pi}{3}]$. The area of the region enclosed by the two curves is therefore

$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \frac{1}{2}\sin x\right)\Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.$$



(2) Calculate the volume of a cylinder inclined at an angle $\theta = \frac{\pi}{6}$ with height 10 and base of radius 4.



SOLUTION: By Cavalieri's Principle, the volume of this thing is the same as the volume of a regular cylinder of height 10. So the volume is $\pi R^2 h = \pi(4)^2(10) = 160\pi$.

Alternatively, the cross-sectional area is at each y-value $\pi(4)^2 = 16\pi$, so the volume is

$$\int_0^{10} 16\pi \, dy = 160\pi.$$

(3) Calculate the volume of the ramp in the figure below in three ways by integrating the area of the cross sections:

(a) perpendicular to the x-axis.

SOLUTION: Cross sections perpindicular to the x-axis are rectangles of width 4 and height $2 - \frac{1}{3}x$. The volume of the ramp is then

$$\int_0^6 4\left(-\frac{1}{3}x + 2\right) dx = \left(-\frac{2}{3}x^2 + 8x\right)\Big|_0^6 = 24.$$

(b) perpendicular to the y-axis.

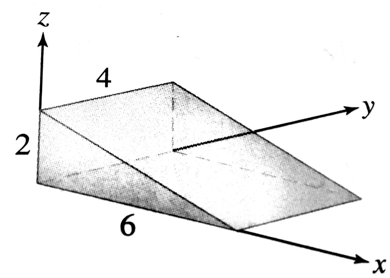
SOLUTION: Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2} \cdot 2 \cdot 6\right) dy = (6y)\Big|_0^4 = 24.$$

(c) perpendicular to the z-axis.

SOLUTION: Cross sections perpendicular to the z-axis are rectangles of length $6 - 3z$ and width 4. The volume of the ramp is then

$$\int_0^2 4(-3(z-2)) \, dz = (-6z^2 + 24z)\Big|_0^2 = 24.$$



(4) Let M be the average value of $f(x) = 2x^2$ on $[0, 2]$. Find a value c such that $f(c) = M$.

SOLUTION: First find the average value

$$M = \frac{1}{2-0}\int_0^2 2x^2\,dx = \frac{2}{2}\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}.$$

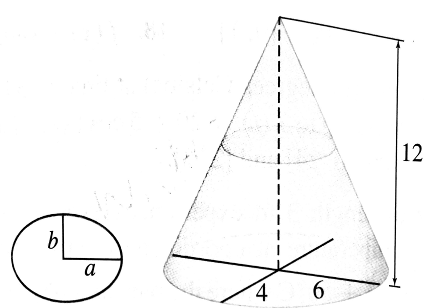
Then $M = f(c) = 2c^2 = \frac{8}{3}$ implies $c = \frac{2}{\sqrt{3}}$.

(5) Find the flow rate through a tube of radius 2 meters, if it's fluid velocity at distance r meters from the center is $v(r) = 4 - r^2$.

SOLUTION: The flow rate is (in meters cubed per second)

$$2\pi\int_0^R rv(r)\,dr = 2\pi\int_0^2 r(4-r^2)\,dr = 2\pi\left(2r^2 - \frac{1}{r}r^4\right)\Big|_0^2 = 8\pi.$$

(6) Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.



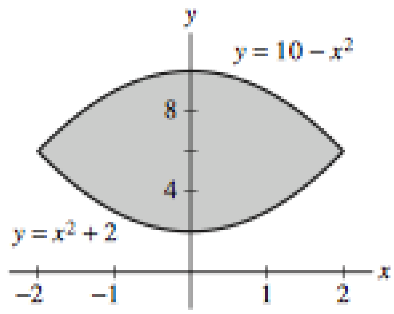
SOLUTION: At each height y, the elliptical cross section has major axis $\frac{1}{2}(12-y)$ and minor axis $\frac{1}{3}(12-y)$. The cross-sectional area is then $\frac{\pi}{6}(12-y)^2$, and the volume is

$$\int_0^{12}\frac{\pi}{6}(12-y)^2\,dy = -\frac{\pi}{18}(12-y)^3\Big|_0^{12} = 96\pi.$$

(7) Sketch the region enclosed by the curves, and determine the cross section perpendicular to the x-axis. Set up an integral for the volume of revolution obtained by rotating the region around the x-axis, but do not evaluate.

(a) $y = x^2 + 2, y = 10 - x^2$.

SOLUTION: Setting $x^2 + 2x = 10 - x^2$ yields $x = \pm 2$. The region enclosed is the figure below.

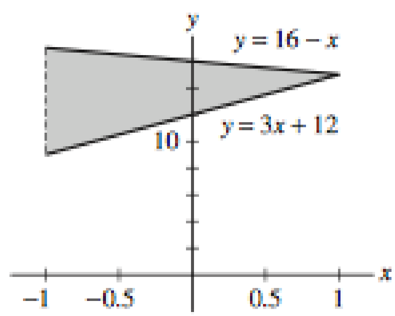


When this is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is then

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2 \right) \, dx$$

(b) $y = 16 - x, y = 3x + 12, x = -1$.

SOLUTION: Setting $16 - x = 3x + 12$ gives $x = 1$. The region enclosed is in the picture below.



When rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$. So the volume of the solid of revolution is

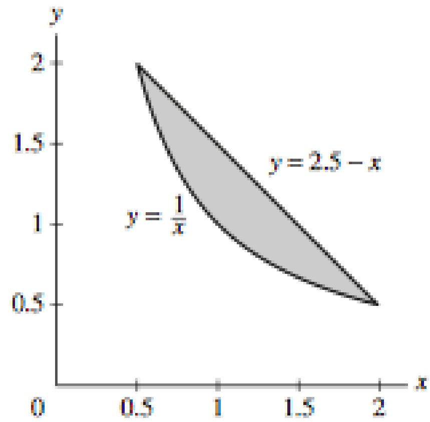
$$\pi \int_{-1}^{1} \left((16 - x)^2 - (3x + 12)^2 \right) \, dx.$$

(c) $y = \frac{1}{x}, y = \frac{5}{2} - x$.

SOLUTION: Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x - 2)(x - \tfrac{1}{2}).$$

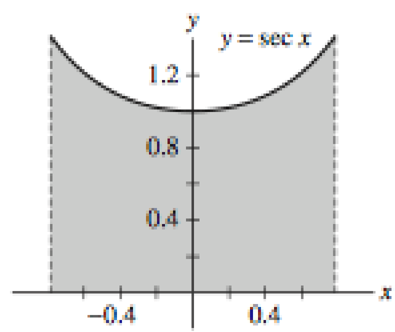
So $x = 2$ and $x = \frac{1}{2}$ are the two points of intersection. The region enclosed by the curves is in the picture below.



The cross sections are washers with outer radius $R = \frac{5}{2} - x$ and inner radius $r = \frac{1}{x}$. So the volume is

$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right) dx$$

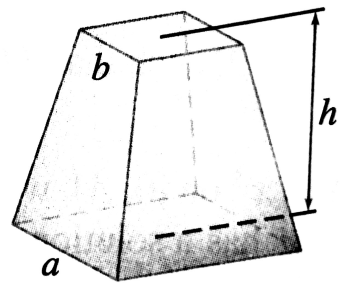
(d) $y = \sec x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$.
SOLUTION: The region in question is



When rotated around the x-axis, each cross section is a circular disk with radius $R = \sec x$. The volume of this solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2 \, dx$$

(8) A frustrum of a pyramid is a pyramid with its top cut off. Let V be the volume of a frustrum of height h whose base is a square of side a and whose top is a square of side b with $a > b > 0$.



(a) Show that if the frustrum were continued to a full pyramid (i.e. the top wasn't cut off), it would have height $ha/(a-b)$.

SOLUTION: Let H be the height of the full pyramid. Using similar triangles, we have the proportion $\frac{H}{a} = \frac{H-h}{b}$, which gives $H = \frac{ha}{a-b}$.

(b) Show that the cross sectional area at height x is a square of side $(1/h)(a(h-x)+bx)$.

SOLUTION: Let w denote the side length of the square cross-section at height x. By similar triangles, we have $\frac{a}{H} = \frac{w}{H-x}$. Substituting H from part (a) gives

$$w = \frac{a(h-x)+bx}{h}.$$

(c) Show that $V = \frac{1}{3}h(a^2+ab+b^2)$.

SOLUTION: The volume of the frustrum is

$$\begin{aligned}\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2 dx &= \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right) dx \\ &= \frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\Bigg|_0^h \\ &= \frac{h}{3}\left(a^2+ab+b^2\right)\end{aligned}$$