§7.7: L'Hôpital's Rule
§7.8 Inverse Trig

Math 1910

Name: Solutions

October 12, 2017

One-page Review

(1) **L'Hôspital's Rule:** If $f(a) = g(a) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} =$ $\boxed{\lim_{x\to a} \frac{f'(x)}{g'(x)}}$.

(2) What are all the indeterminate forms? There are seven of them. $\boxed{\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0}$

(3) To evaluate the limit involving an indeterminate form 0^0, 1^∞, or ∞^0, first take the logarithm and then apply L'Hôspital's rule.

(4) Domain and range of inverse trigonometric functions.

Function	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\cos^{-1}(x)$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$
$\sec^{-1}(x)$	$(-\infty,-1)\cup(1,\infty)$	$[0,\pi/2)\cup(\pi/2,\pi]$
$\csc^{-1}$	$(-\infty,-1)\cup(1,\infty)$	$[-\pi/2,0)\cup(0,\pi/2]$

(5) Derivatives and integrals involving inverse trigonometric functions.

$f(x)$	$\frac{d}{dx}f(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{x^2+1}$
$\cot^{-1}(x)$	$\frac{-1}{x^2+1}$
$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\csc^{-1}(x)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

$f(x)$	$\int f(x)\,dx$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)+C$
$\frac{1}{x^2+1}$	$\tan^{-1}(x)+C$
$\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$	$\sec^{-1}(x)+C$

PROBLEMS

(1) Use L'Hôspital's Rule to calculate the limit

(a) $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7}$

SOLUTION:

$$\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7} = \lim_{x\to\infty} \frac{9x^2+8x}{12x^2} = \lim_{x\to\infty} \left(\frac{9}{12}+\frac{8}{12x}\right) = \frac{3}{4}.$$

(b) $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2}$

SOLUTION: We actually need L'Hôspital's rule for this one! If you plug in $x = 8$ you get the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2} = \lim_{x\to 8} \frac{\frac{5}{3}x^{2/3}-2}{\frac{1}{3}x^{-2/3}} = \lim_{x\to 8}(5x^{4/3}-6x^{2/3}) = 5(8)^{4/3}-6(8)^{2/3} = 56$$

(c) $\lim_{x\to 0} \left(\frac{1}{x^2}-\csc^2 x\right)$

SOLUTION:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x^2}-\csc^2 x\right) &= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2\sin^2 x} \\
&= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2\sin^2 x} \\
&= \lim_{x\to 0} \frac{2\sin x\cos x - 2x}{2x^2\sin x\cos x + 2x\sin^2 x} \\
&= \lim_{x\to 0} \frac{\sin 2x - 2x}{x^2\sin 2x + 2x\sin^2 x} \\
&= \lim_{x\to 0} \frac{2\cos 2x - 2}{2x^2\cos 2x + 2x\sin 2x + 4x\sin x\cos x + 2\sin^2 x} \\
&= \lim_{x\to 0} \frac{\cos 2x - 1}{x^2\cos 2x + 2x\sin 2x + \sin^2 x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{-2x^2\sin 2x + 2x\cos 2x + 4x\cos 2x + 2\sin 2x + 2\sin x\cos x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{(3-2x^2)\sin 2x + 6x\cos 2x} \\
&= \lim_{x\to 0} \frac{-4\cos 2x}{2(3-2x^2)\cos 2x - 4x\sin 2x + -12x\sin 2x + 6\cos 2x} \\
&= -\frac{1}{3}
\end{aligned}$$

(d) $\lim_{x\to\infty} \frac{e^x - e}{\ln x}$

SOLUTION:

$$\lim_{x\to\infty} \frac{e^x - e}{\ln x} = \lim_{x\to\infty} \frac{e^x}{x^{-1}} = \frac{\infty}{0} = \infty$$

(e) $\lim_{x\to\infty} x^{1/x^2}$

SOLUTION: First, compute

$$\lim_{x\to\infty} \ln x^{1/x^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x^2} = \lim_{x\to\infty} e^{\ln x^{1/x^2}} = e^0 = 1.$$

(f) $\lim_{x\to 0^+} x^{\sin x}$

SOLUTION: First, compute

$$\begin{aligned}
\lim_{x\to 0^+} \ln x^{\sin x} &= \lim_{x\to 0^+} \sin x \ln x \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\cos x(\sin x)^{-2}} \\
&= \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x} \\
&= \lim_{x\to 0^+} -\frac{2\sin x\cos x}{-x\sin x + \cos x} = 0
\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\ln x^{\sin x}} = e^0 = 1.$$

(2) Find the derivative.

(a) $y = \arctan(x/3)$

SOLUTION: $y' = \dfrac{1}{(x^2/3)+3}$

(b) $y = \sec^{-1}(x+1)$

SOLUTION: $y' = \dfrac{1}{|x+1|\sqrt{x^2+2x}}$

(c) $y = e^{\cos^{-1}(x)}$

SOLUTION: $y' = \dfrac{-e^{\cos^{-1}(x)}}{\sqrt{1-x^2}}$

(d) $y = \csc^{-1}(x^{-1})$

SOLUTION: $y' = \dfrac{1}{\sqrt{1-x^2}}$

(e) $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

SOLUTION: $y' = \dfrac{1}{t^2+1}$

(f) $y = \frac{\cos^{-1}(x)}{\sin^{-1}(x)}$

SOLUTION: $\dfrac{-\pi}{2\sqrt{1-x^2}(\sin^{-1}(x))^2}$

(g) $y = \cos^{-1}(x + \sin^{-1}(x))$

SOLUTION: $y' = \dfrac{-1}{\sqrt{1 - (x + \sin^{-1} x)^2}} \left(1 + \dfrac{1}{\sqrt{1 - x^2}}\right)$

(h) $y = \ln(\arcsin(x))$

SOLUTION: $y' = \dfrac{1}{\arcsin x \sqrt{1 - x^2}}$

(3) Evaluate the integral

(a) $\displaystyle\int_0^4 \frac{1}{4x^2 + 9}\, dx$

SOLUTION: Let $x = (3/2)u$. Then $dx = (3/2)du$, and $4x^2 + 9 = 9u^2 + 9 = 9(u^2 + 1)$, and

$$\int_0^4 \frac{1}{4x^2+9}\, dx = \frac{1}{6}\int_0^{8/3} \frac{1}{u^2+1}\, du = \frac{1}{6}\tan^{-1} u\Big|_0^{8/3} = \frac{1}{6}\tan^{-1}\left(\frac{8}{3}\right)$$

(b) $\displaystyle\int_{-1/5}^{1/5} \frac{1}{\sqrt{4 - 25x^2}}\, dx$

SOLUTION: Let $x = 2u/5$. Then $dx = \frac{2}{5}du$, and $4 - 25x^2 = 4(1 - u^2)$. So

$$\int_{-1/5}^{1/5} \frac{1}{\sqrt{4-25x^2}}\, dx = \frac{2}{5}\int_{-1/2}^{1/2} \frac{1}{\sqrt{4(1-u^2)}}\, du = \frac{1}{5}\sin^{-1} u\Big|_{-1/2}^{1/2} = \frac{\pi}{12}$$

(c) $\displaystyle\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2 - 1}}\, dx$

SOLUTION: Let $x = u/4$. Then $dx = du/4$, $16x^2 - 1 = u^2 - 1$, and

$$\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2-1}}\, dx = \int_{\sqrt{2}}^{2} \frac{1}{u\sqrt{u^2-1}}\, du = \sec^{-1} u\Big|_{\sqrt{2}}^{2} = \frac{\pi}{12}$$

(d) $\displaystyle\int \frac{1}{x\sqrt{x^4 - 1}}\, dx$

SOLUTION: Let $u = x^2$. Then $du = 2x\, dx$, and

$$\int \frac{1}{x\sqrt{x^4-1}} = \int \frac{1}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1} u + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

(e) $\displaystyle\int \frac{(x+1)}{\sqrt{1 - x^2}}\, dx$

SOLUTION: Observe that

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\, dx = \int \frac{x}{\sqrt{1-x^2}}\, dx + \int \frac{1}{\sqrt{1-x^2}}\, dx$$

In the first integral on the right hand side, we let $u = 1 - x^2$, $du = -2x\, dx$. Then

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\, du + \frac{1}{\sqrt{1-x^2}}\, dx = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

(f) $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

SOLUTION: Let $u = \tan^{-1}(x)$. Then $du = \frac{dx}{1+x^2}$, and

$$\int \frac{\tan^{-1}(x)}{1+x^2}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{(\tan^{-1} x)^2}{2} + C.$$

(g) $\int \frac{1}{\sqrt{5^{2x}-1}}\,dx$

SOLUTION: First, rewrite

$$\int \frac{1}{\sqrt{5^{2x}-1}}\,dx = \int \frac{1}{5^x\sqrt{1-5^{-2x}}} = \int \frac{5^{-x}}{\sqrt{1-5^{-2x}}}$$

Now let $u = 5^{-x}$. Then $du = -5^{-x}\ln 5\,dx$, and

$$\int \frac{1}{\sqrt{5^{2x}-1}} = -\frac{1}{\ln 5}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{\ln 5}\sin^{-1} u + C = -\frac{1}{\ln 5}\sin^{-1}(5^{-x}) + C$$