§7.8: Inverse Trig
§7.9: Hyperbolic Trigonometry
§8.1: Integration by Parts

Math 1910

Name: Solutions

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One-page Review

(1) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\quad\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\quad\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$ $\quad\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$ $\quad\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

(2) Derivatives of hyperbolic trigonometric functions

$$\begin{aligned}
\frac{d}{dx}\sinh(x) &= \cosh(x) & \frac{d}{dx}\cosh(x) &= \sinh(x)\\
\frac{d}{dx}\tanh(x) &= \operatorname{sech}^2(x) & \frac{d}{dx}\coth(x) &= -\operatorname{csch}^2(x)\\
\frac{d}{dx}\operatorname{sech}(x) &= -\operatorname{sech}(x)\tanh(x) & \frac{d}{dx}\operatorname{csch}(x) &= -\operatorname{csch}(x)\coth(x)
\end{aligned}$$

(3) Integrals of hyperbolic trigonometric functions

$$\begin{aligned}
\int \sinh(x)\,dx &= \cosh(x) + C & \int \cosh(x)\,dx &= \sinh(x) + C\\
\int \operatorname{sech}^2(x)\,dx &= \tanh(x) + C & \int \operatorname{csch}^2(x)\,dx &= -\coth(x) + C\\
\int \operatorname{sech}(x)\tanh(x)\,dx &= -\operatorname{sech}(x) + C & \int \operatorname{csch}(x)\coth(x)\,dx &= -\operatorname{csch}(x) + C
\end{aligned}$$

(4) **Integration by parts**

$$\int u\,dv = \boxed{vu - \int v\,du} \quad (1)$$

(5) (Repeat from last Thursday) Derivatives and integrals involving inverse trigonometric functions.

$$\begin{aligned}
\tfrac{d}{dx}\sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}}\\
\tfrac{d}{dx}\cos^{-1}(x) &= \frac{-1}{\sqrt{1-x^2}}\\
\tfrac{d}{dx}\tan^{-1}(x) &= \frac{1}{x^2+1}\\
\tfrac{d}{dx}\cot^{-1}(x) &= \frac{-1}{x^2+1}\\
\tfrac{d}{dx}\sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2+1}}\\
\tfrac{d}{dx}\csc^{-1}(x) &= \frac{-1}{|x|\sqrt{x^2+1}}
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x) + C\\
\int \frac{1}{x^2+1}\,dx &= \tan^{-1}(x) + C\\
\int \frac{1}{|x|\sqrt{x^2+1}}\,dx &= \sec^{-1}(x) + C
\end{aligned}$$

PROBLEMS

(1) Simplify $\sinh(\ln x)$ and $\tanh(\frac{1}{2}\ln(x))$.

SOLUTION: $\sinh(\ln x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ and $\tanh(\dfrac{1}{2}\ln x) = \dfrac{x-1}{x+1}$ for $x > 0$.

(2) Find the derivative.

(a) $y = \ln(\cosh(x))$.

SOLUTION: $y' = \tanh(x)$

(b) $y = \text{sech}(x)\coth(x)$.

SOLUTION: $y' = \text{sech}(x)(-\text{csch}^2(x) - 1)$

(3) Evaluate the integral.

(a) $\displaystyle\int \cosh(2x)\,dx$

SOLUTION: $\dfrac{1}{2}\sinh(2x) + C$

(b) $\displaystyle\int \tanh(3t)\text{sech}(3t)\,dt$

SOLUTION: $-\dfrac{1}{3}\text{sech}(3t) + C$

(c) $\displaystyle\int \frac{\cosh(x)}{3\sinh(x)+4}\,dx$

SOLUTION: $\dfrac{1}{3}\ln|3\sinh(x)+4| + C$

(d) $\displaystyle\int xe^{-x}\,dx$

SOLUTION: Let $u = x$ and $dv = e^{-x}$. Then $u = x$, $du = dx$, and $v = -e^{-x}$. So

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int (1)(-e^{-x})\,dx = -e^{-x}(x+1) + C.$$

(e) $\displaystyle\int x^3 e^{x^2}\,dx$.

SOLUTION: Let $w = x^2$. Then $dw = 2x\,dx$ and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw.$$

Now use integration by parts with $u = w$ and $dv = e^w$. We have $du = 1$ and $v = e^w$, so

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw = we^w - \int (1)e^w\,dw = we^w - e^2.$$

Finally, substitute back $w = x^2$ to get

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

(f) $\int_1^3 \ln x \, dx$.

SOLUTION: Let $u = \ln x$ and $dv = 1$. Then $v = x$ and $du = 1/x$. So using integration by parts,

$$\int_1^3 \ln x \, dx = x \ln x \Big|_1^3 - \int_1^3 1 \, dx = 3 \ln 3 - 2.$$

(4) Find the volume of the solid obtained by revolving $y = \cos x$ for $0 \leq x \leq \pi/2$ around the y-axis.

SOLUTION: Using the cylindrical shells method, the volume V is given by

$$V = \int_a^b (2\pi r) h \, dx = 2\pi \int_0^{\pi/2} x \cos x \, dx.$$

and the radius $r = x$ varies from 0 to $\pi/2$, the height is $h = y = \cos x$. Then using integration by parts, with $u = x$ and $dv = \cos x$, we get

$$V = 2\pi \int_0^{\pi/2} x \cos x \, dx = 2\pi \left(x \sin x + \cos x \right) \Big|_0^{\pi/2} = \pi(\pi - 2).$$

(5) (Repeat from last Thursday) Evaluate the integral.

(a) $\int_0^4 \frac{1}{4x^2 + 9} \, dx$

SOLUTION: Let $x = (3/2)u$. Then $dx = (3/2) du$, and $4x^2 + 9 = 9u^2 + 9 = 9(u^2 + 1)$, and

$$\int_0^4 \frac{1}{4x^2+9} \, dx = \frac{1}{6} \int_0^{8/3} \frac{1}{u^2+1} \, du = \frac{1}{6} \tan^{-1} u \Big|_0^{8/3} = \frac{1}{6} \tan^{-1} \left(\frac{8}{3} \right)$$

(b) $\int_{-1/5}^{1/5} \frac{1}{\sqrt{4 - 25x^2}} \, dx$

SOLUTION: Let $x = 2u/5$. Then $dx = \frac{2}{5} du$, and $4 - 25x^2 = 4(1 - u^2)$. So

$$\int_{-1/5}^{1/5} \frac{1}{\sqrt{4 - 25x^2}} \, dx = \frac{2}{5} \int_{-1/2}^{1/2} \frac{1}{\sqrt{4(1-u^2)}} \, du = \frac{1}{5} \sin^{-1} u \Big|_{-1/2}^{1/2} = \frac{\pi}{12}$$

(c) $\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2 - 1}} \, dx$

SOLUTION: Let $x = u/4$. Then $dx = du/4$, $16x^2 - 1 = u^2 - 1$, and

$$\int_{\sqrt{2}/4}^{1/2} \frac{1}{x\sqrt{16x^2-1}} \, dx = \int_{\sqrt{2}}^{2} \frac{1}{u\sqrt{u^2-1}} \, du = \sec^{-1} u \Big|_{\sqrt{2}}^{2} = \frac{\pi}{12}$$

(d) $\int \frac{1}{x\sqrt{x^4 - 1}} \, dx$

SOLUTION: Let $u = x^2$. Then $du = 2x \, dx$, and

$$\int \frac{1}{x\sqrt{x^4-1}} = \int \frac{1}{2u\sqrt{u^2-1}} = \frac{1}{2} \sec^{-1} u + C = \frac{1}{2} \sec^{-1} x^2 + C.$$

(e) $\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx$

SOLUTION: Observe that

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

In the first integral on the right hand side, we let $u = 1-x^2$, $du = -2x\,dx$. Then

$$\int \frac{(x+1)}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du + \frac{1}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

(f) $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

SOLUTION: Let $u = \tan^{-1}(x)$. Then $du = \frac{dx}{1+x^2}$, and

$$\int \frac{\tan^{-1}(x)}{1+x^2}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{(\tan^{-1} x)^2}{2} + C.$$

(g) $\int \frac{1}{\sqrt{5^{2x}-1}}\,dx$

SOLUTION: First, rewrite

$$\int \frac{1}{\sqrt{5^{2x}-1}}\,dx = \int \frac{1}{5^x\sqrt{1-5^{-2x}}} = \int \frac{5^{-x}}{\sqrt{1-5^{-2x}}}$$

Now let $u = 5^{-x}$. Then $du = -5^{-x}\ln 5\,dx$, and

$$\int \frac{1}{\sqrt{5^{2x}-1}} = -\frac{1}{\ln 5}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{\ln 5}\sin^{-1} u + C = -\frac{1}{\ln 5}\sin^{-1}(5^{-x}) + C$$