§8.7: IMPROPER INTEGRALS
§8.8: PROBABILITY AND INTEGRATION

Math 1910

NAME: SOLUTIONS

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ONE-PAGE REVIEW

(1) The **improper integral** of f over $[a, \infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \boxed{\lim_{R\to\infty} \int_a^R f(x)\,dx.}$$ [1]

We say that the improper integral **converges** if the limit exists [2], and **diverges** if the limit does not exist [3].

(2) The p-**integral** is $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$. If $p > 1$ [4], then this integral converges. If $p \le 1$ [5], then it diverges.

(3) **Comparison test:** Assume $f(x) \ge g(x)$

- If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges [6].
- If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges [7].

(4) If $p(x)$ is a **probability density function** or **PDF**, then $\displaystyle\int_{-\infty}^\infty p(x)\,dx = 1$ [8]

(5) If X is a random variable with probability density function p, then the probability that X is between a and b is

$$P(a \le X \le b) = \boxed{\int_a^b p(x)\,dx}$$ [9]

(6) The **mean** or **average value** of a random variable X with PDF $p(x)$ is $\boxed{\displaystyle\int_{-\infty}^\infty xp(x)\,dx}$ [10].

(7) The **normal distribution** with mean μ and standard deviation σ is the distribution with density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Problems

(1) Determine whether the improper integral converges, and if it does, evaluate it.

(🦇) $\displaystyle\int_1^\infty \frac{1}{x^{20/19}}\,dx$

SOLUTION:

$$\begin{aligned}\int_1^\infty \frac{1}{x^{20/19}}\,dx &= \lim_{a\to\infty}\int_1^a \frac{1}{x^{20/19}}\,dx\\ &= \lim_{a\to\infty}\left(-19x^{-1/19}\right)\Big|_1^a\\ &= \lim_{a\to\infty}\left(-19-\frac{19}{a^{1/19}}\right)\\ &= 19-0=\boxed{19}\end{aligned}$$

(👻) $\displaystyle\int_{20}^\infty \frac{1}{t}\,dt$

SOLUTION: The integral doesn't converge, because it's a p-integral with $p = 1$.

(🎃) $\displaystyle\int_0^5 \frac{1}{x^{19/20}}\,dx$

SOLUTION: The function $x^{-19/20}$ is infinite at the endpoint zero, so it is improper.

$$\begin{aligned}\int_0^5 \frac{1}{x^{19/20}}\,dx &= \lim_{a\to 0}\int_a^5 \frac{1}{x^{19/20}}\\ &= \lim_{a\to 0}\left(20x^{1/20}\right)\Big|_a^5\\ &= \lim_{a\to 0}\left(205^{1/20}-20a^{1/20}\right)\\ &= 20(5^{1/20}-0)=\boxed{20\cdot 5^{1/20}}\end{aligned}$$

(💀) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$

SOLUTION: The function $f(x)=\frac{1}{\sqrt{3-x}}$ is infinite at $x=3$, so it is improper.

$$\begin{aligned}\int_1^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{a\to 3}\int_1^a \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{a\to 3}\left(2\sqrt{3-x}\right)\Big|_1^a\\ &= \lim_{a\to 3} 2\sqrt{3-a}-2\sqrt{2}\\ &= 2\sqrt{0}-2\sqrt{2}=\boxed{2\sqrt{2}}\end{aligned}$$

(🧙) $\displaystyle\int_{-2}^4 \frac{1}{(x+2)^{1/3}}\,dx$

SOLUTION: The function $f(x) = (x+2)^{-1/3}$ is infinite at $x = -2$, so it is improper.

$$
\begin{aligned}
\int_{-2}^{4} \frac{1}{(x+2)^{1/3}}\,dx &= \lim_{a \to -2} \int_{a}^{4} \frac{1}{(x+2)^{1/3}}\,dx \\
&= \lim_{a \to 2} \frac{3}{2}(x+2)^{2/3}\Big|_{a}^{4} \\
&= \lim_{a \to 2} \frac{3}{2}\left(6^{3/2} - (a+2)^{3/2}\right) \\
&= \frac{3}{2}\left(6^{2/3} - 0\right) = \boxed{\frac{3}{2}6^{2/3}}
\end{aligned}
$$

(2) Find a constant C such that $p(x) = {}^{C}/_{(2+x)^3}$ is a probability density function on the interval $[2, 4]$.

SOLUTION: For $p(x)$ to be a probability density function, it must integrate to 1 over the given integral. So

$$
1 = \int_{2}^{4} p(x)\,dx = \int_{2}^{4} \frac{C}{(2+x)^3}\,dx = \frac{-C}{2(2+x)^2}\Big|_{2}^{4} = C\left(\frac{-1}{72} + \frac{1}{32}\right) = C\frac{5}{288}
$$

Therefore, $\boxed{C = \frac{288}{5}}$.

(3) A company produces boxes of rice that are filled on average with 16 oz of halloween candy. Due to the witch's curse placed on the founder of the company many generations ago, the actual volume of candy is normally distributed with a standard deviation of 0.4 oz. Find $P(X > 17)$, the probability of a box having more than 17 oz of candy.

SOLUTION: The problem tells us that the mean is $\mu = 16$ and the standard deviation of $\sigma = 0.4$. We need to find $P(X > 17)$. We have

$$
\begin{aligned}
P(X > 17) &= 1 - P(X \le 17) \\
&= 1 - F\left(\frac{17-16}{0.4}\right) \\
&= 1 - F(2.5) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.5} e^{-t^2/2}\,dt \\
&\approx 0.00621
\end{aligned}
$$