§8.9: NUMERICAL INTEGRATION
§9.1: ARC LENGTH AND SURFACE AREA
§9.4: TAYLOR POLYNOMIALS

NAME: SOLUTIONS

Math 1910

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ONE-PAGE REVIEW

(1) There are three numerical approximations to $\int_a^b f(x)\,dx$:

(a) The **midpoint rule:** $M_N = \Delta x\,(f(c_1) + \ldots + f(c_N))$, $c_j = a + \left(j + \frac{1}{2}\right)\Delta x$.

(b) The **trapezoid rule:** $T_N = \frac{1}{2}\Delta x\,(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N)$

(c) **Simpson's rule:** $S_N = \frac{1}{3}\Delta x\,(y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N)$

(2) The **arc length** of $f(x)$ on the interval $[a, b]$ is $\int_a^b \sqrt{1 + f'(x)^2}\,dx.$ (1)

(3) The **surface area** of the surface obtained by rotating the graph of $f(x)$ around the x-axis for $a \le x \le b$ is $2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$ (2)

(4) The n**-th Taylor Polynomial** centered at $x = a$ for the function f is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$ (3)

(5) The **error for the** n**-th Taylor Polynomial** is

$$|T_n(x) - f(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!},$$ (4)

where K is the maximum of $|f^{(n+1)}(u)|$ over all u between a and x.

(6) **Taylor's Theorem** says that

$$R_n(x) = T_n(x) - f(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,du.$$ (5)

PROBLEMS

(1) Find the T_4 approximation for $\int_0^4 \sqrt{x}\,dx$.

SOLUTION: Let $f(x) = \sqrt{x}$. We divide $[0, 4]$ into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1,$$

with endpoints $0, 1, 2, 3, 4$. With this data, we get

$$T_4 = \frac{1}{2}\Delta x \left(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}\right) \boxed{\approx 5.14626.}$$

(2) State whether M_{10} underestimates or overestimates $\int_1^4 \ln(x)\,dx$.

SOLUTION: Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on the interval $[1, 4]$, so $f(x)$ is concave down. Therefore, the midpoint rule overestimates the integral.

(3) For the curve curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$, set up an integral to calculate:

(a) the arc length.

SOLUTION: First, calculate

$$1 + (y')^2 = 1 + \tan^2(x) = \sec^2(x),$$

so the arc length is

$$\int_0^{\pi/4} \sqrt{1 + (y')^2}\,dx = \int_0^{\pi/4} \sqrt{\sec^2(x)}\,dx = \int_0^{\pi/4} \sec(x)\,dx = \ln|\sec(x) + \tan(x)|\Big|_0^{\pi/4} = \boxed{\ln|\sqrt{2} + 1|}$$

(b) the surface area when rotated around the x-axis.

SOLUTION: As in the previous part, we have

$$1 + (y')^2 = \sec^2(x)$$

Therefore, plug into the arc length formula

$$\text{Surface Area } = 2\pi \int_0^{\pi/4} y\sqrt{1 + (y')^2} = \boxed{2\pi \int_0^{\pi/4} \ln(\cos(x))\sec(x)\,dx}$$

(4) Approximate the arc length of the curve $y = \sin(x)$ over the interval $[0, \pi/2]$ using the midpoint approximation M_8.

SOLUTION: Since $y = \sin(x)$, we have

$$1 + (y')^2 = 1 + \cos^2(x)$$

Therefore, $\sqrt{1+(y')^2} = \sqrt{1+\cos^2(x)}$, and the arc length over $[0, \pi/2]$ is

$$\int_0^{\pi/2} \sqrt{1+\cos^2(x)}\, dx.$$

Let $f(x) = \sqrt{1+\cos^2(x)}$. M_8 is the midpoint approximation with eight subdivisions. So

$$\begin{aligned}
\Delta x &= \frac{\pi/2 - 0}{8} = \frac{\pi}{16} \\
x_i &= 0 + (i - \tfrac{1}{2})\Delta x \qquad \text{for } i = 1, 2, \ldots, 8 \\
y_i &= f\left((i - \tfrac{1}{2})\Delta x\right) \\
M_8 &= \sum_{i=1}^{8} y_i \Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_8)\Delta x
\end{aligned}$$

i	x_i	$f(x_i) = y_i$
1	0.5	1.41081
2	1.5	1.3841
3	2.5	1.3333
4	3.5	1.26394
5	4.5	1.18425
6	5.5	1.10554
7	6.5	1.04128
8	7.5	1.00479

The final answer is that the arc length is approximately $\boxed{1.9101}$.

(5) Find the Taylor polynomials $T_2(x)$ and $T_3(x)$ for $f(x) = \frac{1}{1+x}$ centered at $a = 1$.

SOLUTION: We need to take a few derivatives, and then plug in $a = 1$ to each one.

n	n-th derivative $f^{(n)}(x)$	$f^{(n)}(a)$
0	$f(x) = \dfrac{1}{1+x}$	$f(1) = 1/2$
1	$f'(x) = \dfrac{-1}{(1+x)^2}$	$f'(1) = -1/4$
2	$f''(x) = \dfrac{2}{(1+x)^3}$	$f''(1) = 1/4$
3	$f'''(x) = \dfrac{-6}{(1+x)^4}$	$f'''(1) = -3/8$

Then plug these values into the formula for the Taylor polynomial.

$$T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}$$

$$T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$$

(6) Find n such that $|T_n(1.3) - \sqrt{1.3}| \le 10^{-6}$, where T_n is the Taylor polynomial for $\sqrt{x}$ at $a = 1$.

SOLUTION: By the error formula, we have that

$$|T_n(1.3) - \sqrt{1.3}| \leq \frac{K_{n+1}(1.3-1)^{n+1}}{(n+1)!}$$

So we just need to find n such that

$$\frac{K_{n+1}(0.3)^{n+1}}{(n+1)!} < 10^{-6},$$

where K_{n+1} is the maximum value of the $(n+1)$-st derivative of $f(x) = \sqrt{x}$ between 1 and 1.3. Since $f^{(n+1)}(x)$ is the $(n+1)$-st derivative of $\sqrt{x}$, and this always has x in the denominator for any $n \geq 0$, this maximum will always occur at $x = 1$. Therefore, in this case,

$$K_{n+1} = |f^{(n+1)}(1)|.$$

So we just need to find n such that

$$\frac{|f^{(n+1)}(1)|(0.3)^{n+1}}{(n+1)!} < 10^{-6}.$$

The hard part is finding a pattern for the n-th derivative of $\sqrt{x}$, but that's not strictly necessary, although possible. If you keep taking derivatives of $\sqrt{x}$ and plugging into the formula, you find that this is valid for $\boxed{n \geq 7}$.

Alternatively, the general formula for the n-th derivative of $\sqrt{x}$ is

$$f^{(n)}(x) = (-1)^{n+1}\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{\frac{-(2n-1)}{2}}$$

Then you can plug this in to the previous formula.