§11.1: Sequences
§11.2 Summing an Infinite Series
§11.3 Series with Positive Terms

Math 1910

Name: Solutions

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One-Page Review

(1) If f is continuous and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

(2) A sequences is called:

(a) **bounded** if there exists M such that $|a_n| \leq M$ for all n.

(b) **monotone** if either $a_n < a_{n+1}$ or $a_n > a_{n+1}$ for all n.

If a sequence is both bounded and monotone, then it converges.

(3) **The divergence test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(4) A series that looks like $a_n = cr^n$ is called **geometric.**

(a) If $|r| \geq 1$, then it diverges.

(b) If $|r| < 1$, then $\sum_{n=K}^{\infty} cr^n = \frac{cr^K}{1-r}$

(5) **The integral test:** Assume that $a_n = f(n)$ for $n \geq M$.

(a) If $\int_M^{\infty} f(x)\,dx$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $\int_M^{\infty} f(x)\,dx$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

(6) **The comparison test:**

(a) If $a_n \leq b_n$, and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

(7) **Limit comparison test:** If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is not zero, then $\sum_{n=0}^{\infty} b_n$ converges if and only if $\sum_{n=0}^{\infty} a_n$ converges.

PROBLEMS

(1) True or false?

(a) $\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k$

SOLUTION: True.

(b) $\sum_{n=4}^{6} a_n = \sum_{i=1}^{4} a_{i+3}$

SOLUTION: False.

(c) $\sum_{n=2}^{\infty} a_{n+3} = \sum_{n=5}^{\infty} a_n$

SOLUTION: True

(d) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

SOLUTION: False.

(e) If $\lim_{n\to\infty} a_n = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

SOLUTION: True.

(f) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n = \infty$.

SOLUTION: False.

(2) Determine the limit of the sequence or show that the sequence diverges.

(a) $a_n = \dfrac{e^n}{2^n}$

SOLUTION:

$$a_n = \frac{e^n}{2^n} = \left(\frac{e}{2}\right)^n$$

Note that $e > 2$, so $e/2 > 1$. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{e}{2}\right)^n = \infty.$$

(b) $b_n = \dfrac{3n+1}{2n+4}$

SOLUTION: As $n \to \infty$, the top and the bottom are both polynomial of the same degree, so only the leading coefficients matter. Hence,

$$\lim_{n\to\infty} \frac{3n+1}{2n+4} = \frac{3}{2}.$$

(c) $c_n = \dfrac{\sqrt{n}}{\sqrt{n}+4}$

SOLUTION:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}} + \frac{4}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{1+\frac{4}{\sqrt{n}}} = \frac{1}{1+0} = 1.$$

(3) Show that the sequence given by $a_n = \frac{3n^2}{n^2+2}$ is strictly increasing, and find an upper bound.

SOLUTION: Consider the function $f(x) = \frac{3x^2}{x^2+2}$. The derivative of f is

$$f'(x) = \frac{12x}{(x^2+2)^2}.$$

For $x > 0$, $f'(x) > 0$, so the function is strictly increasing. Therefore, the sequence $a_n = f(n)$ is strictly increasing.

To find an upper bound, observe that

$$a_n = \frac{3n^2}{n^2+2} \leq \frac{3n^2+6}{n^2+2} = \frac{3(n^2+2)}{n^2+2} = 3.$$

Therefore, $M = 3$ is an upper bound.

(4) Determine the limit of the series or show that the series diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$

SOLUTION: This is geometric, and converges to $\frac{1}{1-1/4} = \frac{4}{3}$.

(b) $\displaystyle\sum_{n=0}^{\infty} e^n$

SOLUTION: $\lim_{n\to\infty} e^n = \infty$, so this diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

SOLUTION: This is the Harmonic series, which diverges.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

SOLUTION: This is a telescoping series. First perform partial fractions to see that

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

Then the sum is

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots = 1$$

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$ (Limit Comparison Test)

SOLUTION: Use the limit comparison test. Let $a_n = \frac{n^2}{n^4-1}$. Since for n large, $\frac{n^2}{n^4-1} \approx \frac{n^2}{n^4} = \frac{1}{n^2}$, apply Limit comparison with $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4-1} = 1 \neq 0.$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it's a p-series, so $\sum_{n=2}^{\infty} a_n$ also converges.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2^n}$ (Comparison Test)

SOLUTION: For $n \geq 1$, we have

$$\frac{1}{\sqrt{n}+2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

The series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges since it is geometric with $r = 1/2$. So the comparison test tells us that this series converges too.

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (Integral Test)

SOLUTION: Integrate

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx.$$

Substitute $u = \ln x$, $du = \frac{1}{x}\, dx$. Then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = -\frac{1}{\ln \infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

The integral converges, so the series converges as well.