§11.4: Absolute/Conditional Convergence
§11.5 Ratio and Root Tests

Name: Solutions

Math 1910 — November 21, 2017

Rapid Review

(1) **Absolute Convergence:** A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\boxed{\sum_{n=1}^{\infty} |a_n|}$ (1) converges.

(2) **Absolute Convergence Theorem:** If $\boxed{\sum_{n=1}^{\infty} |a_n|}$ (2) converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) **Conditional Convergence:** A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\boxed{\sum_{n=1}^{\infty} a_n}$ (3) converges but $\boxed{\sum_{n=1}^{\infty} |a_n|}$ (4) diverges.

(4) **Alternating Series Test:** If the sequence $\{b_n\}$ is positive and decreasing, and $\lim_{n\to\infty} b_n = 0$, then $\boxed{\sum_{n=1}^{\infty} (-1)^n b_n}$ (5) converges.

(5) **Ratio Test:** Assume that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists. Then the series $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely when $\boxed{\rho < 1}$ (6)

(b) diverges when $\boxed{\rho > 1}$ (7)

(c) inconclusive if $\boxed{\rho = 1}$ (8)

(6) **Root Test:** Assume that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Then the series $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely if $\boxed{L < 1}$ (9)

(b) diverges if $\boxed{L > 1}$ (10)

(c) inconclusive if $\boxed{L = 1}$ (11)

PROBLEMS

(1) True or false?

(a) If $\sum_{n=0}^{\infty} |b_n|$ diverges, then $\sum_{n=0}^{\infty} b_n$ also diverges.

SOLUTION: False

(b) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges.

SOLUTION: True

(c) If $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} |c_n|$ also converges.

SOLUTION: False

(2) Show that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges conditionally.

SOLUTION: We first show that the series converges, using the Alternating series test. The terms $a_n = \frac{n}{n^2+1}$ tend to zero since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Moreover, a_n is a decreasing sequence because $f(x) = \frac{x}{x^2+1}$ is decreasing for $x \geq 1$. Therefore, the alternating series test applies and the series converges.

However, to show conditional convergence, we have to show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. We can do this with the limit comparison test, comparing a_n to $1/n$. We have

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Therefore, because the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

(3) Does $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$ converges absolutely, conditionally, or not at all?

SOLUTION: Compute the limit

$$\lim_{n\to\infty} \frac{n^4}{n^3+1} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n^3}} = \infty.$$

It follows that the general term $\frac{(-1)^n n^4}{n^3+1}$ of the series doesn't tend to zero, hence this series diverges by the divergence test.

(4) Apply the ratio test or the root test to determine the convergence or divergence of the following series, or state that the test is inconclusive. If the test is inconclusive, apply another test to determine convergence or divergence, if possible.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$

SOLUTION: Use the ratio test. Then $a_n = \frac{(-1)^{n-1} n}{5^n}$. Then $|a_n| = \frac{n}{5^n}$, so compute

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} n + 15n = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{5} = \frac{1}{5}.$$

Since $\rho < 1$, the series converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{5n^3+1}$

SOLUTION: Use the ratio test.

$$\rho = \lim_{n\to\infty} \frac{\frac{3(n+1)+2}{5(n+1)^3+1}}{\frac{3n+2}{5n^3+1}} = \lim_{n\to\infty} \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1} = \lim_{n\to\infty} \frac{15n^4+\dots}{15n^4+\dots} = 1.$$

So $\rho = 1$ and the test is inconclusive.

However, we can use the limit comparison test to compare the series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and see that the series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$

SOLUTION: Use the ratio test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Since $\rho > 1$, the series diverges.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^n}$

SOLUTION: Use the root test.

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{10^n}} = \lim_{n\to\infty} \frac{1}{10} = \frac{1}{10}.$$

Since $L < 1$, this series converges by the root test. Note that this is a geometric series, so we already knew that it converged.

(e) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$

SOLUTION: Use the root test.

$$L = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{k+10}\right)} = \lim_{k\to\infty} \frac{k}{k+10} = 1$$

So the test is inconclusive.