Optimization Practice

April 19, 2017

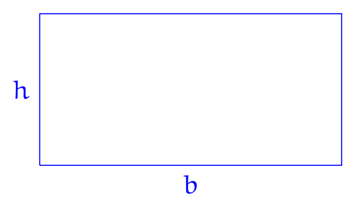
Name: Solutions

Steps for Optimization Problems

1. **Read the problem!** Identify the quantity to be optimized.
2. **Draw a picture** representing the problem. Label any part that is relevant to the problem.
3. **Introduce variables**. List every relation in the picture and in the problem as an equation or expression, and identify the unknown variables.
4. **Write an equation for the quantity you want to optimize**. Use your relations from the previous step to turn it into a function of a single variable. This may require considerable manipulation.
5. **Solve the problem**. Determine the domain of your function. Use the first and second derivative tests to identify and classify the critical points. Check critical points and endpoints to find the optimal value.

(1) Find the dimensions of a rectangle with area of 1,000 square meters whose perimeter is as small as possible.

Solution: If a rectangle has a base of length b meters and a height of length h meters, then the area of the rectangle is $A = bh$ and the perimeter is $P = 2b + 2h$.



We know that $bh = 1000$ and we want to minimize the perimeter P. We can use $bh = 1000$ to substitute for h in the equation $P = 2b + 2h$, so that P is a function of only the variable b:

$$P = 2b + 2h = 2b + 2\left(\frac{1000}{b}\right)$$

Now we want to minimize P. We can do this using the first derivative test.

$$\frac{dP}{db} = 2 - \frac{2000}{b^2}$$

Set this equal to zero and solve for b.

$$0 = 2 - \frac{2000}{b^2} \implies 2b^2 = 2000 \implies b^2 = 1000 \implies b = \sqrt{1000}$$

This value of b minimizes the perimeter. Since the area is $bh = 1000$, we have that $h = \sqrt{1000}$ as well.

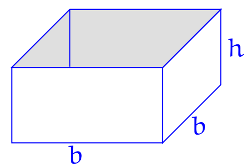
So the dimensions of the rectangle with area 1000 square meters whose perimeter is as small as possible are

$$\boxed{b = \sqrt{1000}, h = \sqrt{1000}}$$

This material is based upon original Active Calculus materials produced by the University of Nebraska at Omaha.

(2) A box with a square base and an open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimize the amount of material used.

SOLUTION:



If the length of one side of the box is b^2 and the height is h, then the volume is $V = b^2h$, and the surface area of the box is the area of the base plus the area of the sides:

$$SA = b^2 + 4bh$$

We know a relation between b and h given by the volume

$$32000 = V = b^2h \implies h = \frac{32,000}{b^2}$$

So substituting this into the surface are equation, we see that

$$SA = \frac{128,000}{b} + b^2$$

To minimize this, take a derivative with respect to b and find the critical points.

$$\frac{d}{db}(SA) = -\frac{128,000}{b^2} + 2b = 0 \implies 2b = \frac{128,000}{b^2} \implies b^3 = 64,000$$

Therefore, $b = 40$ centimeters, and $h = 20$ centimeters.

(3) Find the point on the line $3x + y = 9$ that is closest to the point $(-3, 1)$.

SOLUTION: A point (x, y) on the line $3x + y = 9$ is a distance $D = \sqrt{(x+3)^2 + (y-1)^2}$ from the point $(-3, 1)$; this is the quantity we are trying to minimize. Using

$$3x + y = 9 \implies y = 9 - 3x$$

we can substitute into the equation for d.

$$D = \sqrt{(x+3)^2 + (9-3x-1)^2}$$

Since $\sqrt{x}$ is a monotonic function, to minimize d, it suffices to minimize d^2 instead. In essence, we can get rid of the square root and instead minimize

$$D^2 = (x+3)^2 + (9-3x-1)^2 = 10x^2 - 42x + 73$$

To minimize this, take a derivative and look for the critical points.

$$\frac{d}{dx}\left(D^2\right) = 20x - 42$$

Set this equal to zero and solve

$$0 = 20x - 42 \implies x = 42/20 = 2.1$$

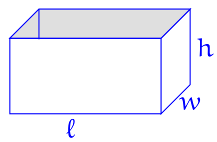
So the x-coordinate of the point on the line $3x + y = 9$ that is closest to $(-3, 1)$ is $x = 2.1$. Plugging this in to $3x + y = 9$, we can solve for y as well:

$$3(2.1) + y = 9 \implies y = 2.7$$

Hence, the coordinates of this point are $(2.1, 2.7)$.

(4) A rectangle storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of the material for the cheapest such container.

SOLUTION:



The volume of the cube is $V = \ell w h$, but $\ell = 2w$, so $V = 2w^2 h$. We are given that $V = 10$ cubic meters, so $2w^2 h = 10$. We may solve for h in terms of w as $h = \frac{10}{w^2}$.

The cost of the box is the cost of the base plus twice the cost of a short side, plus the cost of a long side.

$$C = 10\ell w + 2 \cdot 6 \cdot wh + 2 \cdot 6 \cdot \ell h$$

Substituting $\ell = 2w$:

$$C = 20w^2 + 36wh$$

Now substituting $h = \frac{5}{w^2}$:

$$C = 20w^2 + \frac{180}{w}$$

This is what we want to minimize. So take a derivative with respect to w.

$$\frac{d}{dw} C = 40w - 180w^{-2}$$

Set this equal to zero and solve for w to find the minimizing width.

$$0 = 40w - \frac{180}{w^2} \implies 40w^3 = 180 \implies w^3 = \frac{9}{2}$$

Hence, $\sqrt[3]{9/2} = w$. This is the length that minimizes the cost, but not the actual cost. To find the cost, we must plug w back in to the equation for the cost $C = 20w^2 + 180/w$. Then

$$\boxed{C = 20\left(\frac{9}{2}\right)^{3/2} + \frac{180}{\sqrt[3]{\frac{9}{2}}}}$$