EVALUATING INTEGRALS

May 3, 2017

NAME: SOLUTIONS

The Fundamental Theorem of Calculus, Part I. If f is continuous on $[a, b]$, then for every x in $[a, b]$,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(1) For the following problems, use the Fundamental Theorem of Calculus Part I to find $F'(x)$.

(a) $F(x) = \int_1^x \sqrt[4]{t}\,dt$ SOLUTION: $F'(x) = \sqrt[4]{x}$

(b) $F(x) = \int_x^0 \sec^3 t\,dt$

SOLUTION: $F'(x) = \frac{d}{dx}\int_x^0 \sec^3 t\,dt = \frac{d}{dx}\left(-\int_0^x \sec^3(t)\,dt\right) = -\frac{d}{dx}\int_0^x \sec^3(t)\,dt = -\sec^3(x)$

(c) $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt$. (Don't forget the chain rule!)

SOLUTION: Let $G(x) = \int_2^x \frac{1}{t^3}\,dt$. Then $F(x) = G(x^2)$; now we may apply the chain rule.

$$\frac{d}{dx}F(x) = \frac{d}{dx}G(x^2) = G'(x^2)\cdot 2x$$

So what is $G'(x^2)$? Well, by FTC I, $G'(x) = \frac{1}{x^3}$, so $G'(x^2) = \frac{1}{x^6}$. Therefore,

$$F'(x) = G'(x^2)\cdot 2x = \frac{1}{x^6}\cdot 2x = \frac{2}{x^5}$$

The Fundamental Theorem of Calculus, Part II. If $F(x)$ is an antiderivative for $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(2) Use the Fundamental Theorem of Calculus Part II to evaluate the following integrals.

(a) $\int_0^3 x^3\,dx$ SOLUTION: $\int_0^3 x^3\,dx = \frac{x^4}{4}\Big|_0^3 = \frac{3^4}{4} - \frac{0^4}{4} = \frac{81}{4}$.

(b) $\int_\pi^{3\pi/2} \cos(x)\,dx$ SOLUTION: $\int_\pi^{3\pi/2} \cos(x)\,dx = \sin(3\pi/2) - \sin(\pi) = -1 - 0 = -1$.

(c) $\int_e^{e^2} \frac{1}{x}\,dx$ SOLUTION: $\int_e^{e^2} \frac{1}{x}\,dx = \ln(x)\Big|_e^{e^2} = \ln(e^2) - \ln(e) = 2 - 1 = 1$.

This material is based upon original Active Calculus materials produced by the University of Nebraska at Omaha.

The Substitution Method. To evaluate $\int f(g(x))g'(x)\,dx$:

(1) Substitute $u = g(x)$ and $du = g'(x)\,dx$ to get $\int f(u)\,du$.

(2) Integrate with respect to u.

(3) Replace u by $g(x)$.

(3) Use the substitution method to evaluate the following integrals:

(a) $\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION: Let $u = x^2+1$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$. Hence,

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \frac{1}{2}\cdot -\frac{1}{2}u^{-2}\Big|_1^2 = -\frac{1}{16}+\frac{1}{4} = \frac{3}{16}$$

(b) $\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION: Let $u = x-9$. Then $du = dx$. Hence,

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\Big|_1^8 = 3(2-1) = 3$$

(c) $\int_1^8 \sqrt{t+8}\,dt$

SOLUTION: Let $u = t+8$. Then $du = dt$ and the new bounds are $u = 9$ to $u = 16$.

$$\int_1^8 \sqrt{t+8} = \int_9^{16} \sqrt{u}\,du = u^{3/2}\Big|_9^{16} = 16^{3/2} - 9^{3/2} = 64 - 27 = 37$$

(d) $\int_1^5 \frac{e^x}{3+e^x}\,dx$

SOLUTION: Let $u = 3+e^x$. Then $du = e^x\,dx$, and the bounds become $u = 3+e$ to $u = 3+e^5$. Then

$$\int_1^5 \frac{e^x}{3+e^x}\,dx = \int_{3+e}^{3+e^5} \frac{du}{u} = \ln(u)\Big|_{3+e}^{3+e^5} = \ln(3+e^5) - \ln(3+e) = \ln\left(\frac{3+e}{3+e^5}\right)$$

(e) $\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta$

SOLUTION: Let $u = \cos\theta$; then $du = -\sin\theta\,d\theta$, and the new bounds of integration are $\cos 0 = 1$ to $\cos\pi/2 = 0$. Thus,

$$\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta = -\int_1^0 \sec^2 u\,du = \tan u\Big|_0^1 = \tan 1.$$