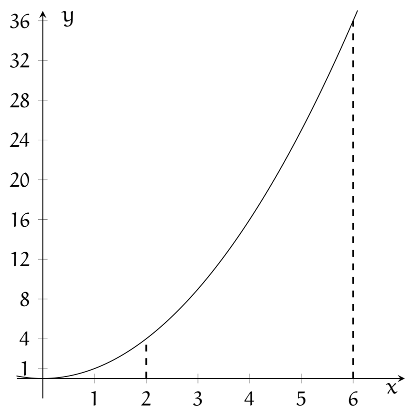
RIEMANN SUMS

NAME: SOLUTIONS

April 26, 2017

Approximating Area Under a Curve



Let's try to approximate the area under the curve $y = x^2$ between $x = 2$ and $x = 6$, as seen in the graph to the left.

We can estimate the area under the curve using rectangles. Let's use four. Since we have four rectangles, the base of each rectangle is 1 unit long, because our interval (from $x = 2$ to $x = 6$) is 4 units long.

So, we have four rectangles with bases in the intervals [2, 3], [3, 4], [4, 5], [5, 6].

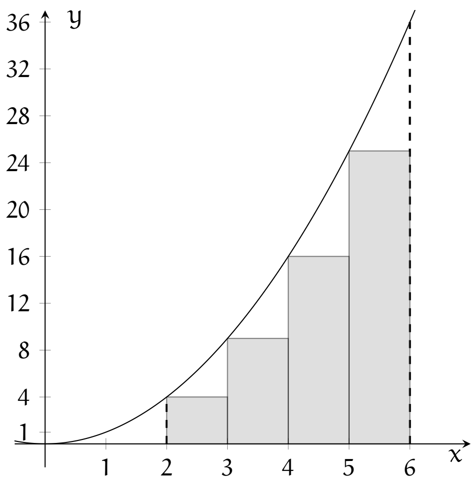
But, where do we get the height of our rectangle from? There are three ways to do this:

(1) Left-handed rectangles

(2) Right-handed rectangles

(3) Midpoint rectangles

Left-Handed Rectangles



When using left-handed rectangles, first evaluate the function at each *left* end point of the intervals.

In our case we have: $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, and $f(5) = 25$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the left end points, as seen in the graph.

(a) Using the four rectangles what is the approximate area?

SOLUTION: The approximate area is the area of the rectangles: the sum of width times height. This is

$$1 * f(2) + 1 * f(3) + 1 * f(4) + 1 * f(5) = 4 + 9 + 16 + 25 = 54$$

(b) Do you think this is an over estimate or an under estimate? Explain.

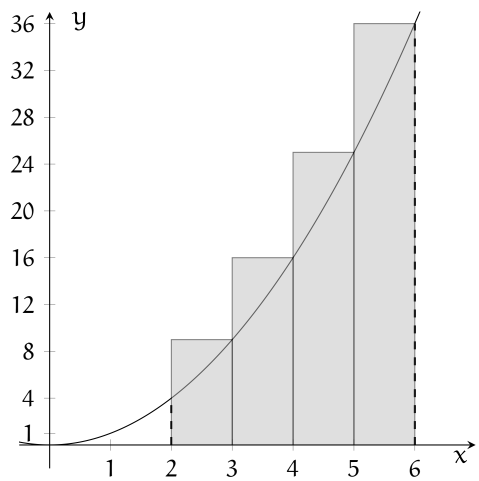
SOLUTION: This is an underestimate: the rectangles clearly don't fill up all the area under the curve.

(c) Write a Riemann sum for this area estimate.

SOLUTION:

$$\sum_{i=2}^{5} (1) * f(i) = \sum_{i=1}^{4} ((i+1) - i) f(i+1)$$

Right-Handed Rectangles



When using right-handed rectangles, first evaluate the function at each *right* end point of the intervals.

In our case we have: $f(3) = 9$, $f(4) = 16$, $f(5) = 25$, and $f(6) = 36$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the right end points, as seen in the graph.

(a) Using the four rectangles what is the approximate area?

SOLUTION: The approximate area is the area of the rectangles: the sum of width times height. This is

$$1 * f(3) + 1 * f(4) + 1 * f(5) + f(6) = 9 + 16 + 25 + 36 = 86$$

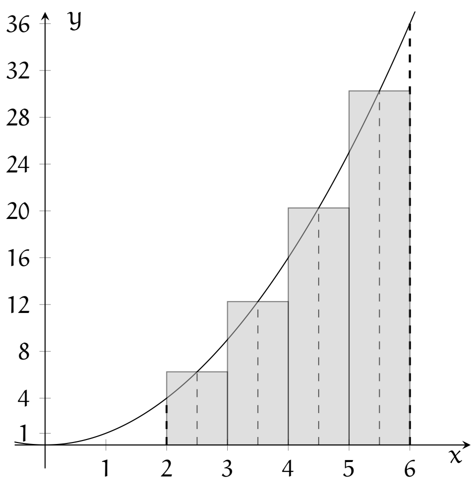
(b) Do you think this is an over estimate or an under estimate? Explain.

SOLUTION: This is clearly an overestimate: the area of the rectangles fills up more than just the area under the curve.

(c) Write a Riemann sum for this area estimate.

$$\sum_{i=2}^{6} (1) f(i+1) = \sum_{k=1}^{4} ((k+1) - k) f(k+2)$$

Midpoint Rectangles



When using midpoint rectangles, first find the *midpoint* of each of the intervals.

In our case, the midpoint of [2, 3] is 2.5, the midpoint of [3, 4] is 3.5, the midpoint of [4, 5] is 4.5, and the midpoint of [5, 6] is 5.5.

Then, evaluate the function at each *midpoint* of the intervals. In our case we have $f(2.5) = 6.25$, $f(3.5) = 12.25$, $f(4.5) = 20.25$, and $f(5.5) = 30.25$. Then, draw a rectangle in each interval with a height equal to that of the value of the function at the midpoint, as seen in the graph.

(a) Using the four rectangles what is the approximate area?

SOLUTION:

$$1 * f(2.5) + 1 * f(3.5) + 1 * f(3.5) + 1 * f(5.5) = 6.25 + 12.25 + 20.25 + 30.25 = 69$$

(b) Do you think this is an over estimate or an under estimate? Explain.

SOLUTION: This is less clear. An argument can be made either way.

(c) Write a Riemann sum for this area estimate.

SOLUTION:

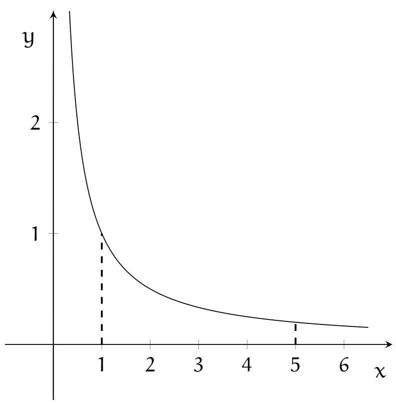
$$\sum_{k=2}^{5} (1) * f(k + 0.5) = \sum_{i=1}^{4} ((i + 1) - i) * f(i + 1.5)$$

What would happen if we had 100 rectangles? 1,000 rectangles? An infinite number of rectangles?

SOLUTION: As the number of rectangles increases, the estimate for the area gets better and better. Therefore, if we had infinite rectangles, we should be able to find the exact area.

This is how we find area under the curve.

Now, you try! Approximate the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 5$ with 4 rectangles, using each of the three methods. The graph is given below.



(a) Write a Riemann sum to estimate this area with 4 midpoint rectangles.

SOLUTION:

$$\sum_{i=1}^{4} (1) * f(i + \tfrac{1}{2})$$

(b) Write a Riemann sum to estimate this area with 8 right-handed rectangles.

SOLUTION:

$$\sum_{i=1}^{8} \tfrac{1}{2} * f(1 + \tfrac{i}{2})$$

(c) Write a Riemann sum to estimate this area with 1000 left-handed rectangles.

SOLUTION:

$$\sum_{i=0}^{999} \tfrac{4}{1000} * f(1 + \tfrac{i*4}{1000})$$

(d) Write a Riemann sum to estimate this area with an arbitrary number of left-handed rectangles, N.

SOLUTION:

$$\sum_{i=0}^{N-1} \tfrac{4}{N} * f(1 + \tfrac{i*4}{N})$$

This material is based upon original Active Calculus materials produced by the University of Nebraska at Omaha.