PROBLEM SET — NAME: SOLUTIONS

§5.2 (Definite Integrals), §5.3 (Indefinite Integrals) — 26 June 2018

(1) Find the following indefinite integrals.

(a) $\int (5x^3 - x^{-2} - x^{3/5})\,dx$ ANSWER: $\frac{5}{4}x^4 + x^{-1} - \frac{5}{8}x^{8/5} + C$

(b) $\int \frac{3}{x^{3/2}}\,dx$ ANSWER: $-\frac{6}{x^{1/2}} + C$

(c) $\int \frac{x^2 + 2x - 3}{x^4}\,dx$ ANSWER: $-x^{-1} - x^{-2} + x^{-3} + C$

(d) $\int 18\cos(3z + 8)\,dz$ ANSWER: $6\sin(3z + 8) + C$

(2) If $f''(x) = x^3 - 2x + 1$, $f'(0) = 0$, and $f(0) = 0$, first find f' and then find f.

ANSWER: $f'(x) = \frac{x^4}{4} - x^2 + x$ and $f(x) = \frac{x^5}{20} - \frac{x^3}{3} + \frac{x^2}{2}$.

(3) Evaluate the sums. (You may use a calculator to do simple arithmetic.)

(a) $\sum_{k=1}^{20} 2k+1$ ANSWER: 440.

(b) $\sum_{j=1}^{10} j^3+2j^2$ ANSWER: 3795.

(c) $\sum_{j=101}^{200} j$ ANSWER: 15050.

(4) Consider the function $f(x) = x^2$ on the interval $[0, 1]$. Find a formula for R_N and compute the area under the graph as a limit. You may use the formula $\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}$.

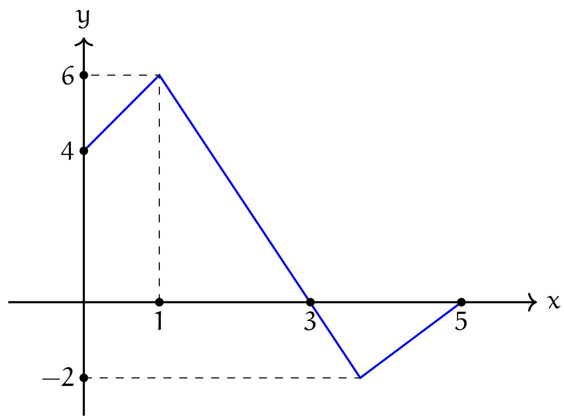
SOLUTION: If we have N rectangles over the interval $[0, 1]$, then the width of each rectangle will be $\Delta x = 1/N$. The height of the i-th rectangle will be $f(i\Delta x) = f(i/N) = i^2/N^2$. So, summing up the area of all of these N rectangles, we find that

$$R_N = \sum_{i=1}^{N} \frac{i^2}{N^2}\frac{1}{N} = \frac{1}{N^3}\sum_{i=1}^{N} i^2 = \frac{1}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right) = \frac{1}{3}+\frac{1}{2N}+\frac{1}{6N^2}.$$

Then, the area under the graph is the limit $\lim_{N\to\infty} R_N$, so we have

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{1}{3}+\frac{1}{2N}+\frac{1}{6N^2}\right) = \boxed{\frac{1}{3}.}$$

(5) Let $f(x)$ be the function plotted below.



Compute the following integrals.

(a) $\int_0^5 f(x)\,dx$ ANSWER: 9

(b) $\int_0^5 |f(x)|\,dx$ ANSWER: 13

(6) Compute the following definite integrals without using the Fundamental Theorem of Calculus. (*Hint: draw a picture.*)

(a) $\int_1^3 |2x-4|\,dx$ ANSWER: 2.

(b) $\int_0^\pi \cos x\,dx$ ANSWER: 0.

(c) $\int_2^6 \sqrt{4-(x-4)^2}\,dx$ ANSWER: 2π.

(7) Recall that a function is called **even** if $f(-x) = f(x)$ for all x, and a function is called **odd** if $f(-x) = -f(x)$ for all x. Explain graphically:

(a) If $f(x)$ is an odd function, $\int_{-a}^{a} f(x)\,dx = 0$.

SOLUTION: If $f(x)$ is an odd function, then its graph is symmetric under rotation about the origin. ($f(x) = x^3$ and $f(x) = \sin(x)$ are examples.) The positive and negative portions of the integral cancel, once we split it at $x = 0$.

(b) If $f(x)$ is an even function $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

SOLUTION: If $f(x)$ is an even function, then its graph is symmetric under rotation reflection across the y-axis. ($f(x) = x^2$ and $f(x) = \cos(x)$ are examples.) The portions of the integral to the left of the y-axis and the right of the y-axis are reflections of one another, so have the same area.

(8) Evaluate $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{j}{N}\right)^2}$ by interpreting the limit as an area.

SOLUTION: The limit represents the area between the graph of $y = f(x) = \sqrt{1 - x^2}$ and the x-axis over the interval $[0, 1]$. This is the portion of the circular disk $x^2 + y^2 \leq 1$ that lies in the first quadrant. Accordingly, its area is $\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$.