§5.4 (FTC I), §5.5 (FTC II)

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NAME: SOLUTIONS

The Fundamental Theorem of Calculus, Part I. If $F(x)$ is an antiderivative for $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(2) Use the Fundamental Theorem of Calculus Part II to evaluate the following integrals.

(a) $\int_0^3 x^3\,dx$ SOLUTION: $\int_0^3 x^3\,dx = \left.\frac{x^4}{4}\right|_0^3 = \frac{3^4}{4} - \frac{0^4}{4} = \frac{81}{4}.$

(b) $\int_\pi^{3\pi/2} \cos(x)\,dx$ SOLUTION: $\int_\pi^{3\pi/2} \cos(x)\,dx = \sin(3\pi/2) - \sin(\pi) = -1 - 0 = -1.$

(c) $\int_e^{e^2} \frac{1}{x}\,dx$ SOLUTION: $\int_e^{e^2} \frac{1}{x}\,dx = \ln(x)\Big|_e^{e^2} = \ln(e^2) - \ln(e) = 2 - 1 = 1.$

The Fundamental Theorem of Calculus, Part II. If f is continuous on $[a, b]$, then for every x in $[a, b]$,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(1) For the following problems, use the Fundamental Theorem of Calculus Part I to find $F'(x)$.

(a) $F(x) = \int_1^x \sqrt[4]{t}\,dt$ SOLUTION: $F'(x) = \sqrt[4]{x}$

(b) $F(x) = \int_x^0 \sec^3 t\,dt$

SOLUTION: $F'(x) = \frac{d}{dx}\int_x^0 \sec^3 t\,dt = \frac{d}{dx}\left(-\int_0^x \sec^3(t)\,dt\right) = -\frac{d}{dx}\int_0^x \sec^3(t)\,dt = -\sec^3(x)$

(c) $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt$. (Don't forget the chain rule!)

SOLUTION: Let $G(x) = \int_2^x \frac{1}{t^3}\,dt$. Then $F(x) = G(x^2)$; now we may apply the chain rule.

$$\frac{d}{dx}F(x) = \frac{d}{dx}G(x^2) = G'(x^2)\cdot 2x$$

So what is $G'(x^2)$? Well, by FTC I, $G'(x) = \frac{1}{x^3}$, so $G'(x^2) = \frac{1}{x^6}$. Therefore,

$$F'(x) = G'(x^2)\cdot 2x = \frac{1}{x^6}\cdot 2x = \frac{2}{x^5}$$

(d) $F(x) = \int_{-x}^{3x} \sqrt{t^2+1}\,dt$

SOLUTION:

$$\begin{aligned}
\frac{d}{dx}\int_{-x}^{3x}\sqrt{t^2+1}\,dt &= \frac{d}{dx}\left(\int_{-x}^{0}\sqrt{t^2+1}\,dt + \int_0^{3x}\sqrt{t^2+1}\,dt\right)\\
&= \frac{d}{dx}\left(-\int_0^{-x}\sqrt{t^2+1}\,dt\right) + \frac{d}{dx}\int_0^{3x}\sqrt{t^2+1}\,dt\\
&= \frac{d}{dx}\left(F(3x) - F(-x)\right)\\
&= F'(x)\cdot 3 - F'(x)\cdot(-1)\\
&= 3\sqrt{9x^2+1} + \sqrt{x^2+1}
\end{aligned}$$