§5.6 (Rate of Change) — Name: Solutions

28 July 2018

(1) Find the displacement over the time interval $[1, 6]$ of a helicopter whose vertical velocity at time t is $v(t) = .02t^2 + t$ feet per second.

Solution: Given $v(t) = \frac{1}{50}t^2 + t$ feet per second, the change in height over $[1, 6]$ is

$$\int_1^6 v(t)\,dt = \int_1^6 \left(\frac{1}{50}t^2 + t\right)\,dt$$
$$= \left(\frac{1}{150}t^3 + \frac{1}{2}t^2\right)\Big|_1^6$$
$$= \left(\frac{1}{150}6^3 + \frac{1}{2}6^2\right) - \left(\frac{1}{150}1^3 + \frac{1}{2}1^2\right) = \frac{284}{15} \approx 18.93\text{feet.}$$

(2) A particle is moving along a straight line with velocity $v(t) = \cos t$ meters per second. Find

(a) the total displacement over the interval $[0, 4\pi]$, and

Solution: Total displacement is given by $\int_0^{4\pi} \cos t\,dt = \sin t\Big|_0^{4\pi} = 0$ meters.

(b) the total distance travelled over the interval $[0, 4\pi]$.

Solution: Total distance is given by

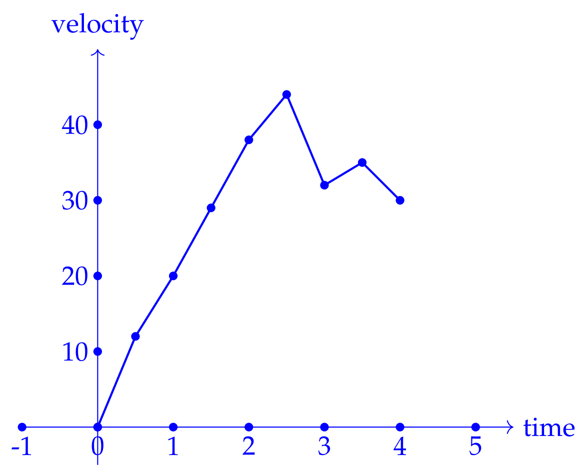
$$\int_0^{4\pi} |\cos t|\,dt = \int_0^{\pi/2} \cos t\,dt + \int_{\pi/2}^{3\pi/2} -\cos t\,dt + \int_{3\pi/2}^{5\pi/2} \cos t\,dt + \int_{5\pi/2}^{7\pi/2} -\cos t\,dt + \int_{7\pi/2}^{4\pi} \cos t\,dt$$
$$= \sin t\Big|_0^{\pi/2} - \sin t\Big|_{\pi/2}^{3\pi/2} + \sin t\Big|_{3\pi/2}^{5\pi/2} - \sin t\Big|_{5\pi/2}^{7\pi/2} + \sin t\Big|_{7\pi/2}^{4\pi}$$
$$= 8 \text{ meters}$$

(3) The velocity in feet per second of a car is recorded at half-second intervals in the table below.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	0	12	20	29	38	44	32	35	30

Use the average of the left-endpoint and right-endpoint approximations to estimate the total distance travelled over the time interval $[0, 4]$.

SOLUTION: First, draw a picture!



Let $\Delta x = 0.5$. Then

$$R_N = 0.5 \cdot (12 + 20 + 29 + 38 + 44 + 32 + 35 + 30) = 120 \text{ feet}$$

$$L_N = 0.5 \cdot (0 + 12 + 20 + 29 + 38 + 44 + 32 + 35) = 105 \text{ feet}$$

The average of R_N and L_N is 112.5 feet, which is a decent estimate for the distance travelled by the car over the interval.

(4) The heat capacity $C(T)$ of a substance is the amount of energy (in joules) required to raise the temperature of one gram of the substance by one degree Celsius when it's temperature is T. (The heat capacity depends on the substance's current temperature.)

(a) Determine the energy required to raise the temperature of one gram from T_1 to T_2

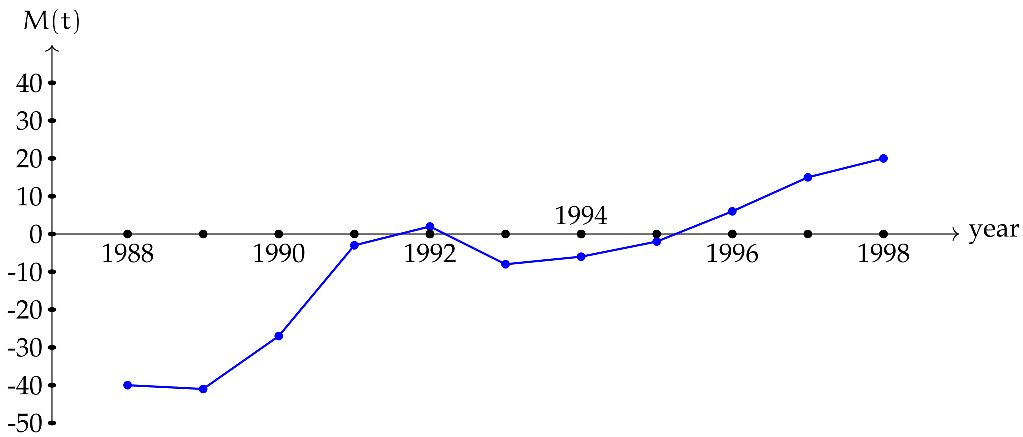
SOLUTION: Since $C(T)$ is the energy required to raise the temperature of one gram by one degree, when the temperature is T, the energy required to raise the temperature depends on the current temperature, which is always changing as energy is added. Hence, the total energy needed is the area under the graph of $C(T)$ between T_1 and T_2, which is $\int_{T_1}^{T_2} C(T)\,dT$.

(b) If a substance has heat capacity $C(T) = 6 + 0.2\sqrt{T}$, calculate the energy required to raise the temperature of one gram of the substance from 50° to 100° Celsius.

SOLUTION: If $C(T) = 6 + 0.2\sqrt{T} = 6 + \frac{1}{5}T^{1/2}$, then the energy required to raise the temperature from 50° C to 100° C is

$$\begin{aligned}
\int_{50}^{100} C(T)\,dT &= \int_{50}^{100} \left(6 + \frac{1}{5}t^{1/2}\right)\,dt \\
&= \left(6t + \frac{2}{15}t^{3/2}\right)\Bigg|_{50}^{100} \\
&= \left(6(100) + \frac{2}{15}(100)^{3/2}\right) - \left(6(50) + \frac{2}{15}(50)^{3/2}\right) \\
&= \frac{1300 - 100\sqrt{2}}{3} \approx 386.19 \text{ Joules}
\end{aligned}$$

(5) The migration rate $M(t)$ of Ireland in the period 1988-1998 is shown in the figure below. This is the rate at which people (in thousands of people per year) move into or out of the country.



(a) Is the following integral positive or negative? What does the quantity represent? $\int_{1988}^{1998} M(t)\,dt$

SOLUTION: The integral is negative: there is clearly more area beneath the year axis than above it. The quantity represents the total number of people that moved into or out of Ireland over this ten year period.

(b) Did migration in the period 1988 – 1998 result in a net influx of people into Ireland or a net outflow of people from Ireland?

SOLUTION: A net outflow, based on the answer to the previous question.

(c) During which two years could the Irish prime minister announce: "We're still losing population, but the trend is now improving?"

SOLUTION: We are looking for inflection points on the graph. These occur in 1993 and in 1989.

§5.7 (SUBSTITUTION) NAME:________________________

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The Substitution Method. To evaluate $\int f(g(x))g'(x)\,dx$:

(1) Substitute $u = g(x)$ and $du = g'(x)\,dx$ to get $\int f(u)\,du$.

(2) Integrate with respect to u.

(3) Replace u by $g(x)$.

(6) Use the substitution method to evaluate the following integrals:

(a) $\displaystyle\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION: Let $u = x^2+1$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$. Hence,

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \frac{1}{2}\cdot -\frac{1}{2}u^{-2}\Big|_1^2 = -\frac{1}{16}+\frac{1}{4} = \frac{3}{16}$$

(b) $\displaystyle\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION: Let $u = x-9$. Then $du = dx$. Hence,

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\Big|_1^8 = 3(2-1) = 3$$

(c) $\displaystyle\int_1^8 \sqrt{t+8}\,dt$

SOLUTION: Let $u = t+8$. Then $du = dt$ and the new bounds are $u = 9$ to $u = 16$.

$$\int_1^8 \sqrt{t+8} = \int_9^{16} \sqrt{u}\,du = u^{3/2}\Big|_9^{16} = 16^{3/2} - 9^{3/2} = 64-27 = 37$$

(d) $\displaystyle\int_1^5 \frac{e^x}{3+e^x}\,dx$

SOLUTION: Let $u = 3+e^x$. Then $du = e^x\,dx$, and the bounds become $u = 3+e$ to $u = 3+e^5$. Then

$$\int_1^5 \frac{e^x}{3+e^x}\,dx = \int_{3+e}^{3+e^5} \frac{du}{u} = \ln(u)\Big|_{3+e}^{3+e^5} = \ln(3+e^5) - \ln(3+e) = \ln\left(\frac{3+e}{3+e^5}\right)$$

(e) $\displaystyle\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta$

SOLUTION: Let $u = \cos\theta$; then $du = -\sin\theta\,d\theta$, and the new bounds of integration are $\cos 0 = 1$ to $\cos\pi/2 = 0$. Thus,

$$\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\,d\theta = -\int_1^0 \sec^2 u\,du = \tan u\Big|_0^1 = \tan 1.$$

(f) $\displaystyle\int_0^{\pi/4} \tan^3\theta\sec^2\theta\,d\theta$

SOLUTION: Let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$. Then $u^3\,du = \tan^3\theta\sec^2\theta\,d\theta$. The new limits of integration are from $u(0) = \tan 0 = 0$ to $u(\pi/4) = \tan(\pi/4) = 1$. Therefore,

$$\int_0^{\pi/4} \tan^3\theta\sec^2\theta\,d\theta = \int_0^1 u^3\,du = \frac{1}{4}.$$

(g) $\displaystyle\int \frac{dx}{(2+\sqrt{x})^3}$

SOLUTION: Let $u = 2+\sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so that

$$2\sqrt{x}\,du = dx \implies 2(u-2)\,du = dx.$$

Using this, we get

$$\begin{aligned}
\int \frac{dx}{(2+\sqrt{x})^3} &= \int 2\frac{u-2}{u^3}\,du \\
&= 2\int (u^{-2} - 2u^{-3})\,du \\
&= 2(-u^{-1} + u^{-2}) + C \\
&= 2\left(-\frac{1}{2+\sqrt{x}} + \frac{1}{(2+\sqrt{x})^2}\right) + C \\
&= 2\left(\frac{-2-\sqrt{x}+1}{(2+\sqrt{x})^2}\right) + C \\
&= -2\frac{1+\sqrt{x}}{(2+\sqrt{x})^2} + C
\end{aligned}$$