§6.1 (AREA BETWEEN CURVES) NAME: SOLUTIONS

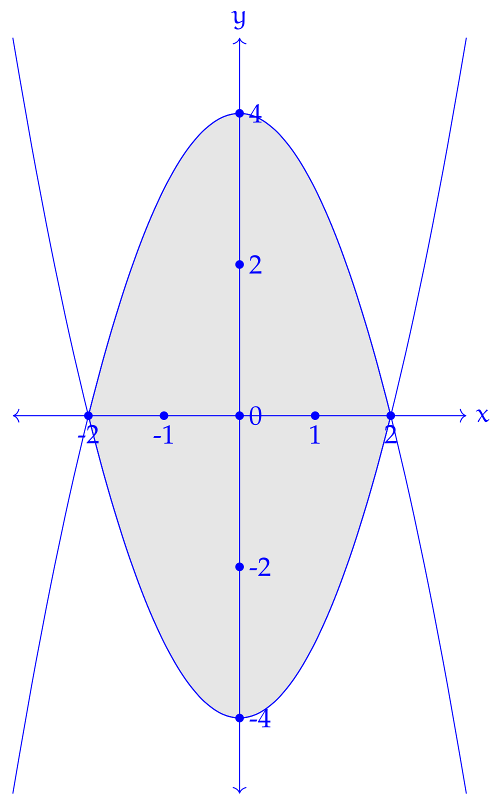
2 July 2018

(1) Sketch the region enclosed by the curves and set up an integral to compute it's area, but do not evaluate.

(a) $y = 4 - x^2, y = x^2 - 4$

SOLUTION: Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4)\right)\, dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left(8x - \frac{2}{3}x^3\right)\Big|_{-2}^{2} = \frac{64}{3}.$$



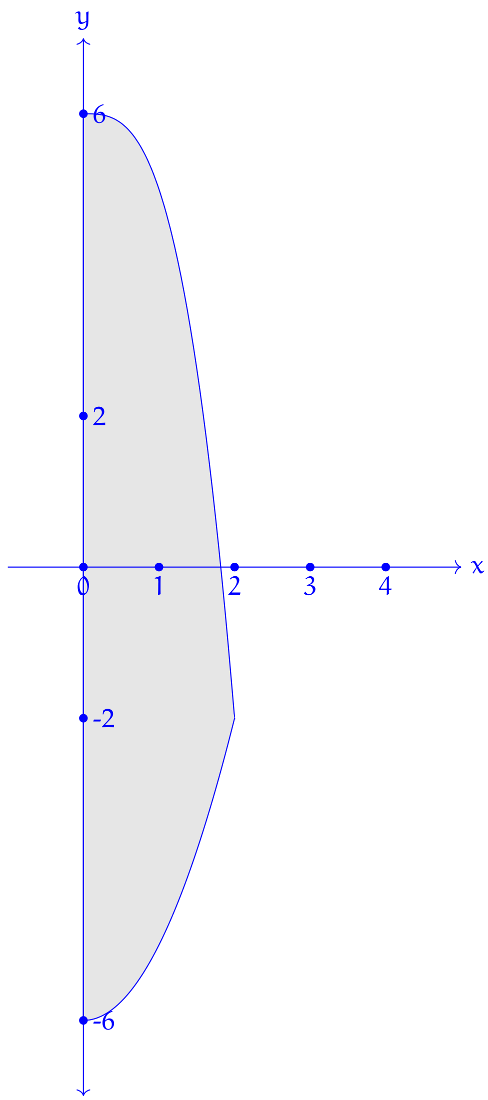
(b) $y = x^2 - 6, y = 6 - x^3, x = 0$

SOLUTION: Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x-2)(x^2+3x+6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval $[0, 2]$; hence, the area of the region enclosed by these curves and the y-axis is

$$\int_0^2 \left((6-x^3)-(x^2-6)\right)\, dx = \int_0^2 (-x^3 - x^2 + 12)\, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x\right)\Big|_0^2 = \frac{52}{3}$$



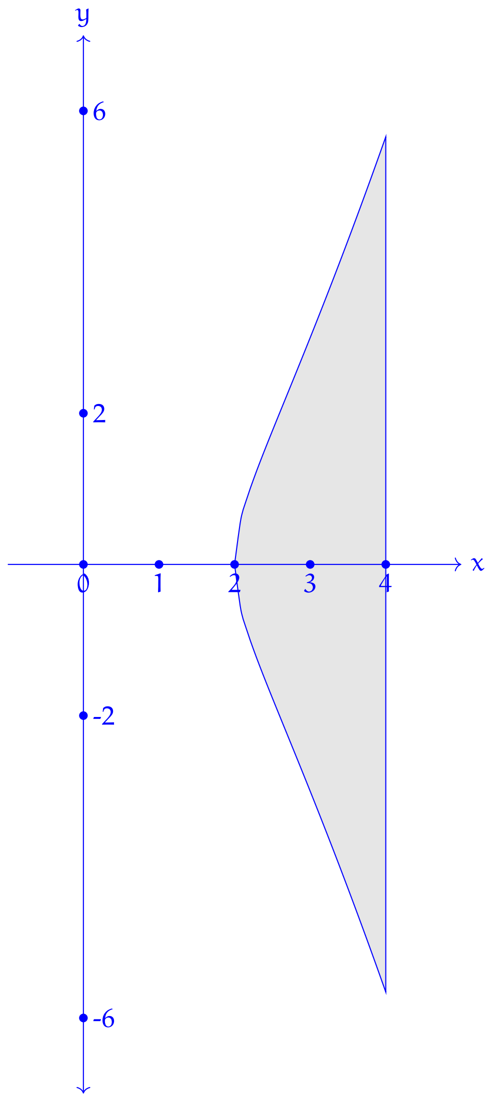
(c) $y = x\sqrt{x-2}$, $y = -x\sqrt{x-2}$, $x = 4$

SOLUTION: Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x-2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_2^4 \left(x\sqrt{x-2} - (-x\sqrt{x-2}) \right) \, dx = \int_2^4 2x\sqrt{x-2}\, dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

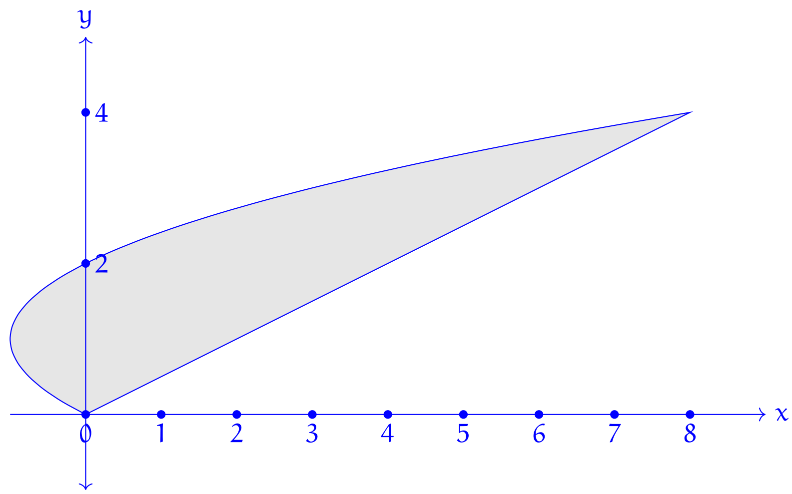
$$\int_2^4 2x\sqrt{x-2}\, dx = \int_0^2 2(u+2)\sqrt{u}\, du = \int_0^2 \left(2u^{3/2} + 4u^{1/2} \right) \, du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2} \right) \Big|_0^2 = \frac{128\sqrt{2}}{15}.$$



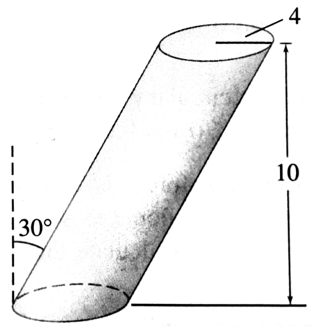
(d) $x = 2y$, $x + 1 = (y-1)^2$

SOLUTION: Setting $2y = (y-1)^2 - 1$ yields $0 = y^2 - 4y = y(y-4)$, so the two curves intersect at $y = 0$ and $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right)\, dy = \int_0^4 (4y - y^2)\, dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



(1) Calculate the volume of a cylinder inclined at an angle $\theta = \frac{\pi}{6}$ with height 10 and base of radius 4.



SOLUTION: By Cavalieri's Principle, the volume of this thing is the same as the volume of a regular cylinder of height 10. So the volume is $\pi R^2 h = \pi(4)^2(10) = 160\pi$.

Alternatively, the cross-sectional area is at each y-value $\pi(4)^2 = 16\pi$, so the volume is

$$\int_0^{10} 16\pi \, dy = 160\pi.$$

(2) Calculate the volume of the ramp in the figure below in three ways by integrating the area of the cross sections:

(a) perpendicular to the x-axis.

SOLUTION: Cross sections perpindicular to the x-axis are rectangles of width 4 and height $2 - \frac{1}{3}x$. The volume of the ramp is then

$$\int_0^6 4\left(-\frac{1}{3}x + 2\right)\, dx = \left(-\frac{2}{3}x^2 + 8x\right)\Big|_0^6 = 24.$$

(b) perpendicular to the y-axis.

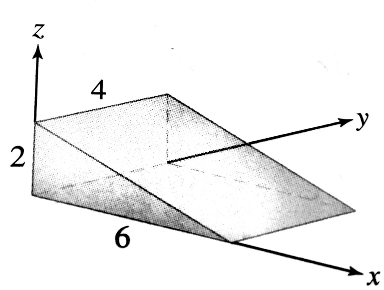
SOLUTION: Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2}\cdot 2 \cdot 6\right)\, dy = (6y)\Big|_0^4 = 24.$$

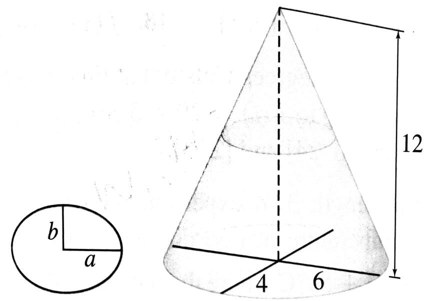
(c) perpendicular to the z-axis.

SOLUTION: Cross sections perpendicular to the z-axis are rectangles of length $6 - 3z$ and width 4. The volume of the ramp is then

$$\int_0^2 4(-3(z-2))\, dz = (-6z^2 + 24z)\Big|_0^2 = 24.$$



(3) Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.



SOLUTION: At each height y, the elliptical cross section has major axis $\frac{1}{2}(12-y)$ and minor axis $\frac{1}{3}(12-y)$. The cross-sectional area is then $\frac{\pi}{6}(12-y)^2$, and the volume is

$$\int_0^{12} \frac{\pi}{6}(12-y)^2\, dy = -\frac{\pi}{18}(12-y)^3 \Big|_0^{12} = 96\pi.$$