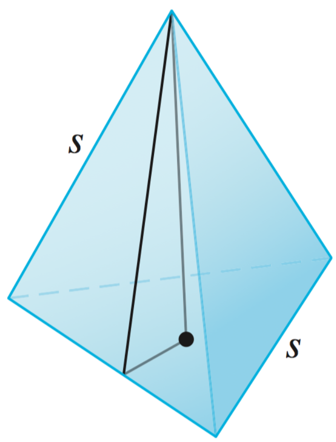
§6.2 (SETTING UP INTEGRALS) NAME: SOLUTIONS

3 July 2018

(1) Find the volume of a *regular* tetrahedron whose faces are equilateral triangles of side length s.



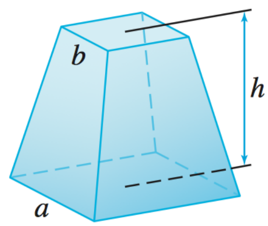
SOLUTION: Using similar triangles, the total height of the tetrahedron is $h = \sqrt{2/3} \cdot s$. Also using similar triangles, the side length of the equilateral triangle at height z above the base is

$$s\left(\frac{h-z}{h}\right) = s - \frac{z}{\sqrt{2/3}}.$$

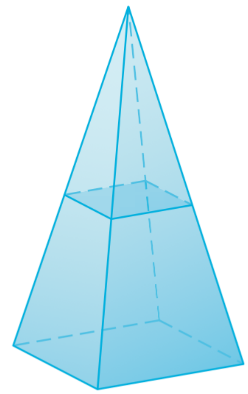
The volume of the tetrahedron is then given by

$$\int_0^{s\sqrt{2/3}} \frac{\sqrt{3}}{4}\left(s - \frac{z}{\sqrt{2/3}}\right)^2 dz = -\frac{\sqrt{2}}{12}\left(s - \frac{z}{\sqrt{2/3}}\right)^3 \Big|_0^{s\sqrt{2/3}} = \frac{s^3\sqrt{2}}{12}.$$

(2) A frustum of a pyramid is a pyramid with its top cut off. Let V be the volume of a frustum of height h whose base is a square of side length a and whose top is a square of side length b with $a > b \geq 0$.



(A)



(B)

(a) Show that if the frustum were continued to a full pyramid, it would have height $\dfrac{ha}{a-b}$.

SOLUTION: Let H be the height of the full pyramid. Using similar triangles, we have the proportion

$$\frac{H}{a} = \frac{H-h}{b},$$

which gives

$$H = \frac{ha}{a-b}.$$

(b) Calculate the side length of a cross-section of the frustum at height x from the base.

SOLUTION: The cross-section at height x is a square of side length $(1/h)(a(h-x)+bx)$.

Let w denote the side length of the square cross-section at height x. By similar triangles, we have

$$\frac{a}{H} = \frac{w}{H-x}.$$

Substituting the value for H from part (a) gives

$$w = \frac{a(h-x)+bx}{h}.$$

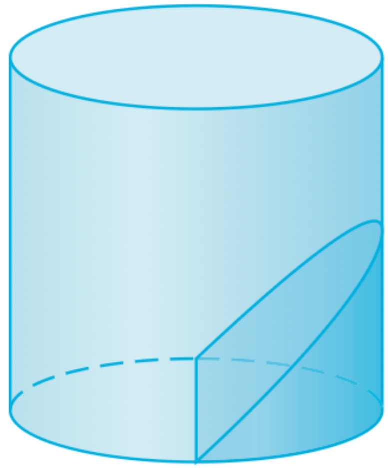
(c) Calculate the volume of the frustum.

SOLUTION: $V = \frac{1}{3}h(a^2+ab+b^2)$.

This volume is obtained by the integral:

$$\begin{aligned}\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2 dx &= \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right) dx \\ &= \frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\Bigg|_0^h \\ &= \frac{h}{3}\left(a^2+ab+b^2\right)\end{aligned}$$

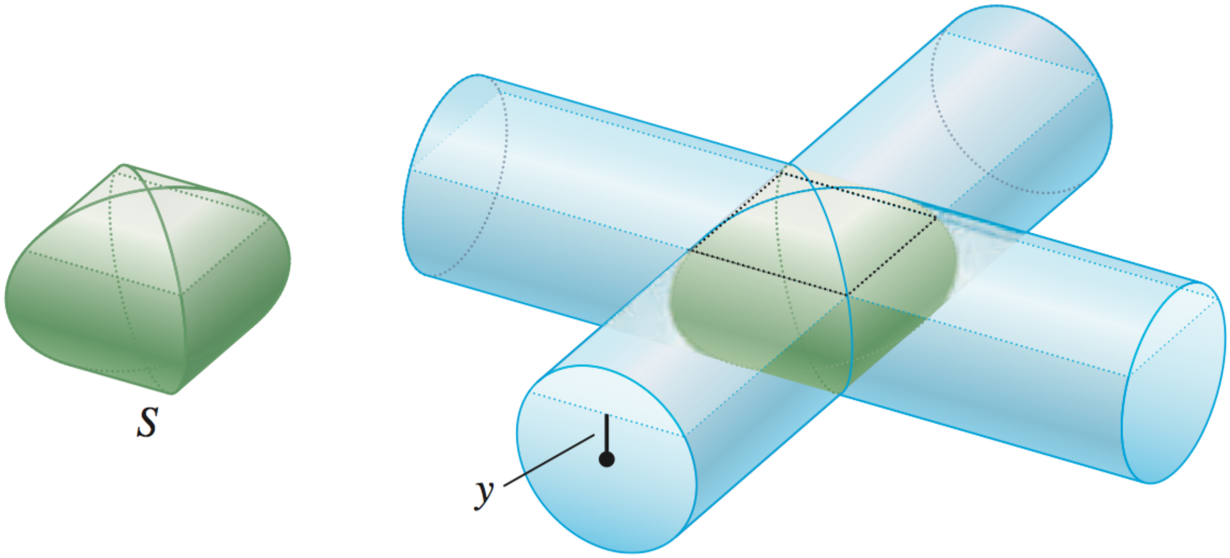
(3) A plane inclined at an angle of $45°$ passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane.



SOLUTION: Place the center of the base at the origin. Then, for each x, the vertical cross section taken perpendicular to the x-axis is a rectangle of base $2\sqrt{r^2 - x^2}$ and height x. The volume of the solid enclosed by the plane and the cylinder is therefore

$$\int_0^r 2x\sqrt{r^2 - x^2}\,dx = \int_0^{r^2} \sqrt{u}\,du = \left(\frac{2}{3}u^{3/2}\right)\Big|_0^{r^2} = \frac{2}{3}r^3.$$

(4) The solid S below is the intersection of two cylinders of radius r whose axes are perpendicular.



(a) The horizontal cross-section of each cylinder at a distance y from the central axis is a rectangular strip. Find the area of the horizontal cross-section of S at distance y from the central axis.

SOLUTION: The horizontal cross section at distance y from the central axis (for $-r \leq y \leq r$) is a square of width $w = 2\sqrt{r^2 - y^2}$. The area of the horizontal cross section of S at distance y from the central axis is $w^2 = 4(r^2 - y^2)$.

(b) Find the volume of S as a function of r.

SOLUTION: The volume of the solid S is then

$$4\int_{r}^{-r} (r^2 - y^2)\, dy = 4\left(r^2 y - \frac{1}{3}y^3\right)\Big|_{-r}^{r} = \frac{16}{3}r^3$$