VOLUMES AND AREAS NAME: SOLUTIONS

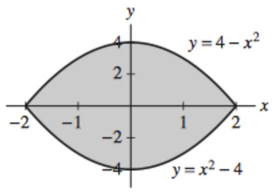
6 July 2018

(1) Sketch the region enclosed by the curves and set up an integral to compute it's area, but do not evaluate.

(a) $y = 4 - x^2, y = x^2 - 4$

SOLUTION: Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4) \right) \, dx = \int_{-2}^{2} (8 - 2x^2) \, dx = \left(8x - \frac{2}{3}x^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



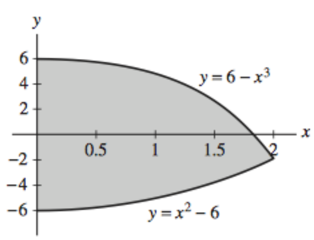
(b) $y = x^2 - 6, y = 6 - x^3, x = 0$

SOLUTION: Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval $[0, 2]$; hence, the area of the region enclosed by these curves and the y-axis is

$$\int_{0}^{2} \left((6 - x^3) - (x^2 - 6) \right) \, dx = \int_{0}^{2} (-x^3 - x^2 + 12) \, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x \right) \Big|_{0}^{2} = \frac{52}{3}$$



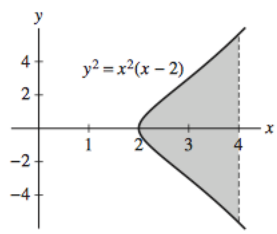
(c) $y = x\sqrt{x-2}, y = -x\sqrt{x-2}, x = 4$

SOLUTION: Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x-2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_2^4 \left(x\sqrt{x-2} - (-x\sqrt{x-2})\right)\, dx = \int_2^4 2x\sqrt{x-2}\, dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

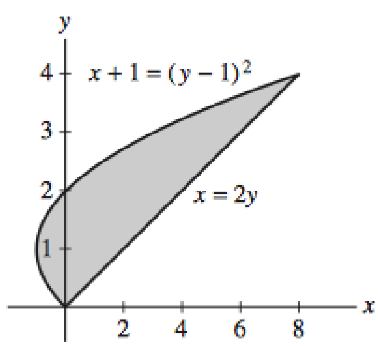
$$\int_2^4 2x\sqrt{x-2}\, dx = \int_0^2 2(u+2)\sqrt{u}\, du = \int_0^2 \left(2u^{3/2} + 4u^{1/2}\right)\, du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right)\Big|_0^2 = \frac{128\sqrt{2}}{15}.$$



(d) $x = 2y, x + 1 = (y-1)^2$

SOLUTION: Setting $2y = (y-1)^2 - 1$ yields $0 = y^2 - 4y = y(y-4)$, so the two curves intersect at $y = 0$ and $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

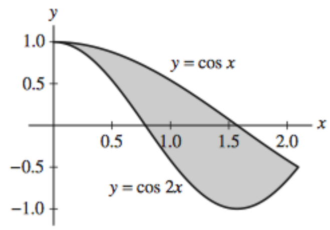
$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right)\, dy = \int_0^4 (4y - y^2)\, dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



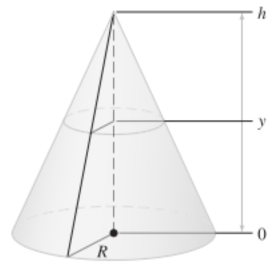
(e) $y = \cos x$, $y = \cos(2x)$, $x = 0$, $x = \frac{2\pi}{3}$

SOLUTION: From the graph below, we see that $y = \cos x$ lies above $y = \cos 2x$ over the interval $[0, \frac{2\pi}{3}]$. The area of the region enclosed by the two curves is therefore

$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \frac{1}{2} \sin x \right) \Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.$$



(2) Let V be the volume of a right circular cone of height h whose base is a circle of radius R. Find its volume V.



SOLUTION: Take horizontal cross sections; if r denotes the radius at a height y, then by similar triangles

$$\frac{h}{R} = \frac{h-y}{r}$$

thus, $r = R - \frac{Ry}{h}$. Finally,

$$V = \pi \int_0^h \left(R - \frac{Ry}{h}\right)^2 dy = \frac{\pi R^2 h}{3}$$

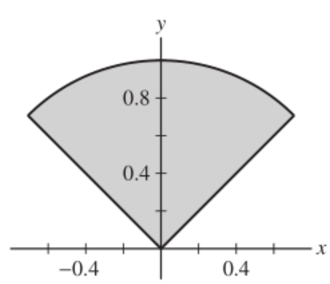
This solution assumes the $y-$axis is placed on the height of the cone, pointing upwards, starting from 0 in the base of the cone. A slightly different solution can be found by placing 0 on the top of the cone and point the $y-$axis downwards; of course both reach the same final answer for the volume.

(3) *Going the other way.* Sketch a region whose area is represented by

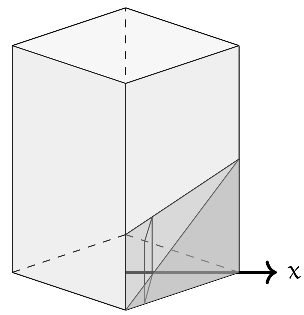
$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\sqrt{1-x^2} - |x|)\,dx$$

and evaluate it using geometry.

SOLUTION: We want our region to be bounded along the top by $y = \sqrt{1-x^2}$, and bounded along the bottom by $y = |x|$. Hence, the region is one quarter of the unit circle, and so $A = \pi/4$.

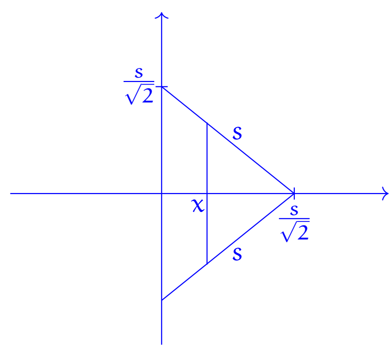


(4) Consider the following box, with square top and bottom lids of side s. A plane inclined at an angle of 45 degrees passes through a diagonal of the bottom lid. Find the volume of the region within the box and below the plane.



SOLUTION: If we place the center of the base at the origin, then the vertical cross section at a given x will be a rectangle of height x, since the 45 degree plane has equation $x = z$.

To find the length of the rectangle, it suffices to look at the bottom triangle and find the length of the line shown in the picture, which is the base of the rectangle seen from above:



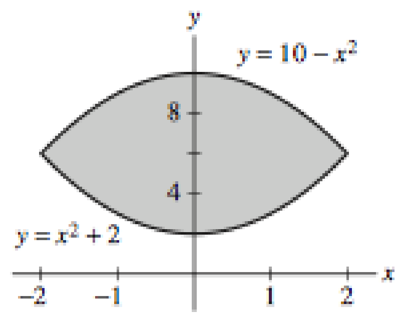
Using geometry (for example, similar triangles), we find that the length is $2(\frac{s}{\sqrt{2}} - x)$, so the volume of the shaded solid is

$$V = 2\int_0^{s/\sqrt{2}} x\left(\frac{s}{\sqrt{2}} - x\right)\,dx = \frac{s^3}{6\sqrt{2}}$$

(5) Sketch the region enclosed by the curves, and determine the cross section perpendicular to the x-axis. Set up an integral for the volume of revolution obtained by rotating the region around the x-axis, but do not evaluate.

(a) $y = x^2 + 2, y = 10 - x^2$.

SOLUTION: Setting $x^2 + 2x = 10 - x^2$ yields $x = \pm 2$. The region enclosed is the figure below.

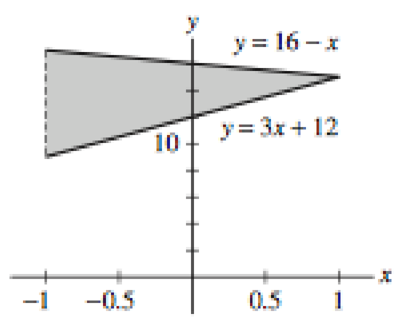


When this is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is then

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2 \right) \, dx$$

(b) $y = 16 - x, y = 3x + 12, x = -1$.

SOLUTION: Setting $16 - x = 3x + 12$ gives $x = 1$. The region enclosed is in the picture below.



When rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$. So the volume of the solid of revolution is

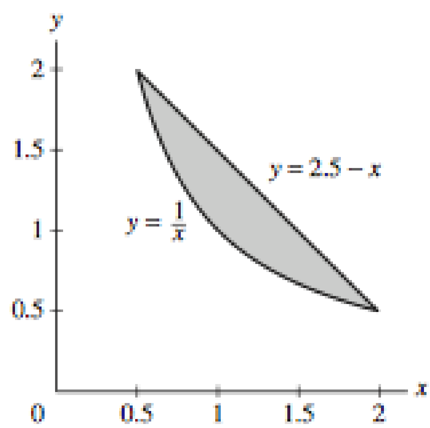
$$\pi \int_{-1}^{1} \left((16 - x)^2 - (3x + 12)^2 \right) \, dx.$$

(c) $y = \frac{1}{x}, y = \frac{5}{2} - x$.

SOLUTION: Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x-2)(x - \tfrac{1}{2}).$$

So $x = 2$ and $x = \frac{1}{2}$ are the two points of intersection. The region enclosed by the curves is in the picture below.

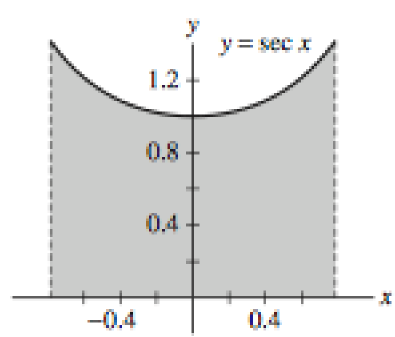


The cross sections are washers with outer radius $R = \frac{5}{2} - x$ and inner radius $r = \frac{1}{x}$. So the volume is

$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right) dx$$

(d) $y = \sec x, y = 0, x = -\frac{\pi}{4}, x = \frac{\pi}{4}$.

SOLUTION: The region in question is



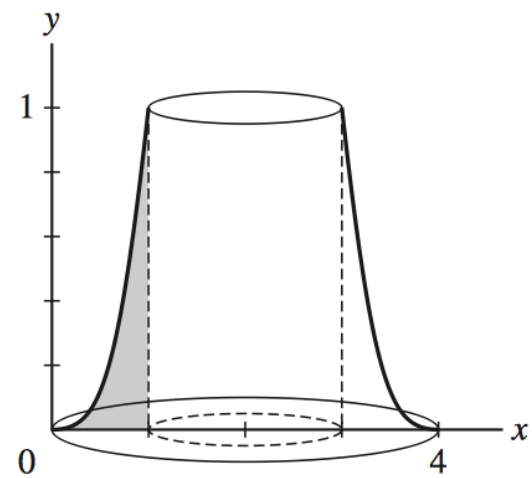
When rotated around the x-axis, each cross section is a circular disk with radius $R = \sec x$. The volume of this solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2 \, dx$$

(6) Sketch the solid obtained by rotating the region underneath the graph of f over the interval about the given axis, and calculate its volume using the shell method.

(a) $f(x) = x^3$, $x \in [0, 1]$, about $x = 2$.

SOLUTION:

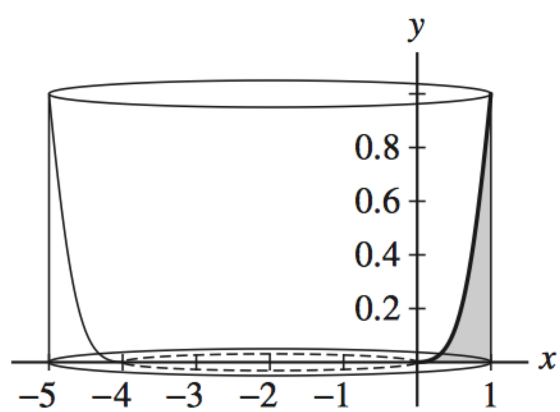


Each shell has radius $2 - x$ and height x^3, so the volume of this solid is

$$2\pi \int_0^1 (2 - x)(x^3)\, dx = 2\pi \int_0^1 (2x^3 - x^4)\, dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{2} \right) \Big|_0^1 = \frac{3\pi}{5}.$$

(b) $f(x) = x^3$, $x \in [0, 1]$, about $x = -2$.

SOLUTION:

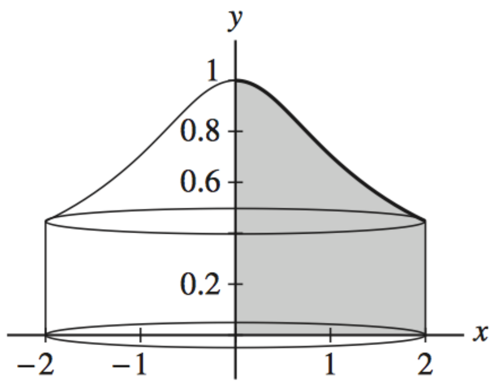


Each shell has radius $x - (-2) = x + 2$ and height x^3, so the volume of this solid is

$$2\pi \int_0^1 (2 + x)(x^3)\, dx = 2\pi \int_0^1 (2x^3 + x^4)\, dx = 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{7\pi}{5}.$$

(c) $f(x) = \frac{1}{\sqrt{x^2+1}}$, $x \in [0, 2]$, about $x = 0$.

SOLUTION:



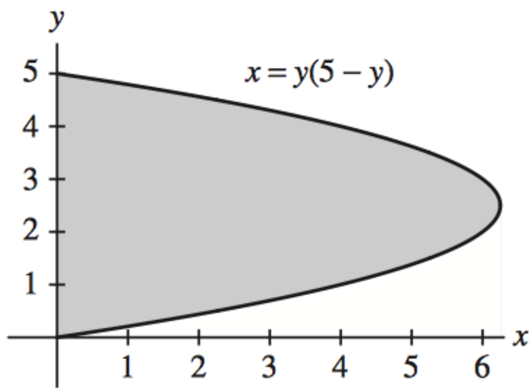
Each shell has radius x, and height $\frac{1}{\sqrt{x^2+1}}$, so the volume of the solid is

$$2\pi \int_0^2 x \left(\frac{1}{\sqrt{x^2+1}} \right) dx = 2\pi \sqrt{x^2+1} \Big|_0^2 = 2\pi \left(\sqrt{5} - 1 \right).$$

(7) Use the most convenient method (disk/washer or shell) to find the given volume of rotation.

(a) Region between $x = y(5-y)$ and $x = 0$, rotated about the y-axis.

SOLUTION: Examine the picture below, which shows the region in question. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizonally, then the right endpoint of a slice is always along the parabola and the left endpoint is on the y-axis. So it's easier to do horizontal slices.

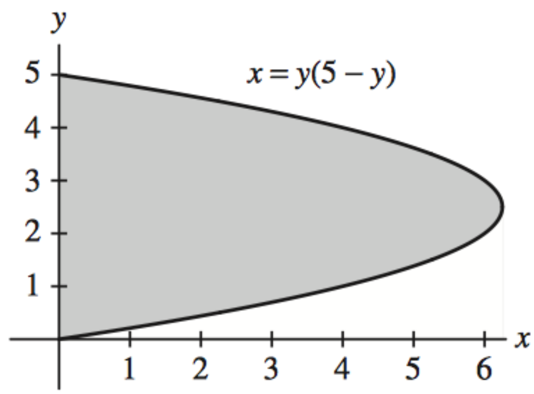


Now suppose the region is rotated about the y-axis. Because a horizontal slice is perpendicular to the y-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = y(5-y)$, so the volume is

$$\pi \int_0^5 y^2(5-y)^2 \, dy = \pi \int_0^5 (25y^2 - 10y^3 + y^4) \, dy = \pi \left(\frac{25}{3}y^3 - \frac{5}{2}y^4 + \frac{1}{5}y^5 \right) \Bigg|_0^5 = \frac{625\pi}{6}.$$

(b) Region between $x = y(5 - y)$ and $x = 0$, rotated around the x-axis.

SOLUTION: Examine the ?gure below, which shows the region bounded by $x = y(5?y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5 - y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

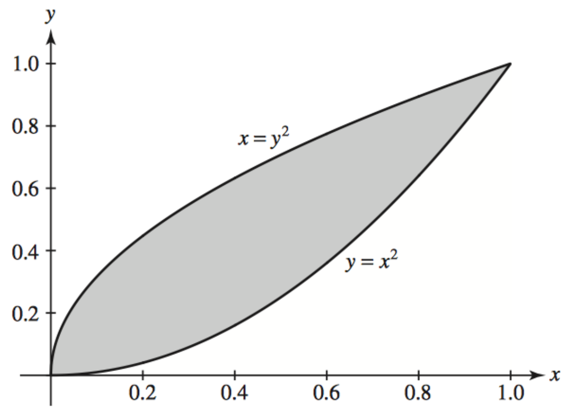


Now, suppose the region is rotated about the x-axis. Because a horizontal slice is parallel to the x-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of y and a height of $y(5 - y)$, so the volume is

$$2\pi \int_0^5 y^2(5 - y)\, dy = 2\pi \int_0^5 (5y^2 - y^3)\, dy = 2\pi \left(\frac{5}{3}y^3 - \frac{1}{4}y^4 \right) \Big|_0^5 = \frac{625\pi}{6}.$$

(c) Region between $y = x^2$ and $x = y^2$, rotated about $x = 3$.

SOLUTION: Examine the figure below, which shows the region bounded by the $y = x^2$ and $x = y^2$. If the indicated region is sliced vertically, then the top of the slice lies along $x = y^2$ and the bottom lies along $y = x^2$. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along $y = x^2$ and left endpoint always lies along $x = y^2$. Thus, for this region, either choice of slice will be convenient. To proceed, let's choose a vertical slice.

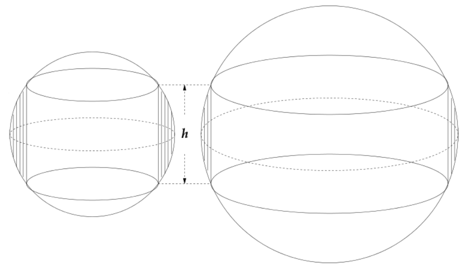


Now rotate the region about $x = 3$. Because a vertical slice is parallel to $x = 3$, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of $3 - x$ and a height of $\sqrt{x} - x^2$, so the volume is

$$2\pi \int_0^1 (3 - x)(\sqrt{x} - x^2)\, dx = 2\pi \int_0^1 (3x^{1/2} - x^{3/2} - 3x^2 + x^3)\, dx$$

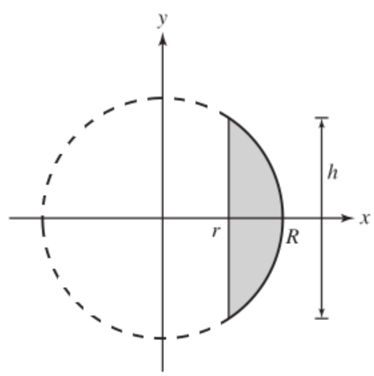
$$= 2\pi \left(2x^{3/2} - \frac{2}{5}x^{5/2} - x^3 + \frac{1}{4}x^4 \right) \Big|_0^1 = \frac{17\pi}{10}.$$

(8) *The napkin-ring problem.* Show that when a hole of height h is drilled straight through the center of a sphere, the volume of the remaining band does not depend on the size of the sphere.



This means that a napkin ring of height 1cm built from a marble will have the same volume as one built from the Sun!

SOLUTION: Consider a sphere of radius R, to which we drill a hole of height h. Then, we can obtain the napkin ring by rotating the shaded region around the $y-$axis:



The (horizontal) cross sections will be washers, and to find the outer and inner radius at a given height y we need to determine the two functions that delimit the shaded region.

As we can see, the outer function is part of the circle of equation $x^2 + y^2 = R^2$, so for a given y, the outer radius will be $R_y = \sqrt{R^2 - y^2}$. The second delimiting function is a constant, and from the Pythagorean Theorem we find that $r = \sqrt{R^2 - (h/2)^2}$.

Therefore, the volume of the napkin ring will be

$$V = \pi \int_{-h/2}^{h/2} ((\sqrt{R^2 - y^2})^2 - (\sqrt{R^2 - h^2/4})^2) dy = \pi \int_{-h/2}^{h/2} \frac{h^2}{4} - y^2 dy = \frac{\pi h^3}{6}$$

which is independent of R, the radius of the sphere.