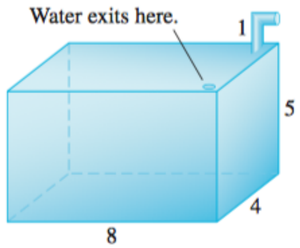
§6.5 (WORK AND ENERGY) NAME: SOLUTIONS

9 July 2018

(1) Calculate the work (in Joules) required to pump all of the water out of a full tank with the shape described. Distances are in meters, and the density of water is 1000 kg/m^3.

(a) A rectangular tank, with water exiting from a small hole in the top.



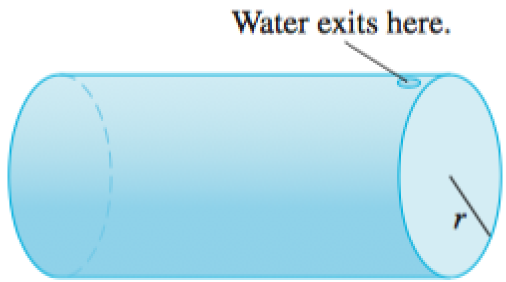
SOLUTION: Place the origin at the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ cubic meters, so the force needed to lift it is

$$(9.8)(1000)(32)\Delta y = 313600\Delta y \text{ Newtons.}$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_0^5 313600y \, dy = 156800y^3 \Big|_0^5 = 3.92 \times 10^6 \text{ Joules.}$$

(b) A horizontal cylinder of length ℓ, where water exits from a small hole in the top.



SOLUTION: Place the origin along the central axis of the cylinder. At location y, the layer of water is a rectangular lab of length ℓ, width $2\sqrt{r^2 - y^2}$, and thickness Δy. Thus, the volume of the layer is $2\ell\sqrt{r^2 - y^2}\Delta y$, and the force needed to lift the layer is

$$19600\ell\sqrt{r^2 - y^2}\Delta y.$$

The layer must be lifted a distance $r - y$, so the total work needed to empty the tank is given by

$$\int_{-r}^{r} 19600\ell\sqrt{r^2 - y^2}(r - y)\,dy = 19600\ell r \int_{-r}^{r} \sqrt{r^2 - y^2}\,dy - 19600 \int_{-r}^{r} y\sqrt{r^2 - y^2}\,dy$$

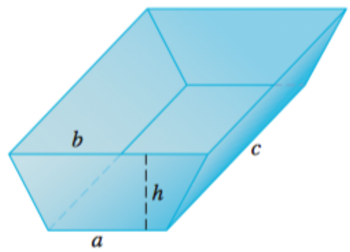
Now the second term is zero because the integrand is an odd function and the interval of integration is symmetric with respect to zero. Moreover, the otehr integral is one-half the area of a circle of radius r, therefore,

$$\int_{-r}^{r} \sqrt{r^2 - y^2}\,dy = \frac{1}{2}\pi r^3.$$

So the total work needed to empty the tank is

$$19600\ell r \left(\frac{1}{2}\pi^2\right) = 9800\ell\pi r^3 \text{ Joules.}$$

(c) A trough as in the picture, where the water exits by pouring over the sides.



SOLUTION: Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at height y meters is

$$w = a + \frac{y(b-a)}{h} \text{ meters,}$$

so the volume of each layer is

$$wc\Delta y = c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ meters}^3.$$

Thus, the force needed to lift a layer is

$$9800c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ Newtons.}$$

Each layer must be lifted $h-y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y(b-a)}{h}\right)dy = 9800c\left(\frac{ah^3}{3} + \frac{bh^2}{6}\right) \text{ Joules.}$$

(2) Calculate the work required to lift a 6 meter chain with mass 18 kg over the side of a building.

SOLUTION: First, note that the chain has a mass density of $18/6 = 3$ kg/m. Now, consider a segment of the chain of length Δy located at distance y_j feet from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (3\Delta y)9.8y_j \text{ Newtons.}$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^6 29.4y\,dy = 14.7y^2\Big|_0^6 = 529.2 \text{ Joules.}$$

(3) A 3 meter chain with mass density $\rho(x) = 2x(4-x)$ kg/m lies on the ground. Calculate the work required to lift the chain from the front end so that its bottom is 2 meters above the ground.

SOLUTION: Consider a segment of the chain of length Δx that must be lifted x_j meters. The work needed to lift this segment is approximately

$$W_j \approx (\rho(x_j)\Delta x)9.8x_j \text{ Joules.}$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work needed to fully extend the chain is

$$\int_0^3 9.8\rho(x)x\,dx = 9.8\int_0^3 (8x^2 - 2x^3)\,dx = 9.8\left(\frac{8}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^3 = 308.7 \text{ Joules.}$$

But we also need to lift the chain two meters off the ground after it's fully extended! This requires us to do work equal to 2 meters multiplied by the weight of the chain, which is

$$\int_0^3 9.8\rho(x)\,dx = 9.8\int_0^3 (8x - 2x^2)\,dx = 9.8\left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 176.4 \text{ Newtons.}$$

So lifting it another two meters after it's fully extended requires an additional 352.8 Joules of work. The total work is therefore 661.5 Joules.