§6.5 (Work and Energy) — Name: Solutions

10 July 2018

(1) Calculate the work required to lift a 3-meter chain over the side of a building if the chain has variable density $\lambda(x) = x^2 - 3x + 10$ kg/m for $0 \leq x \leq 3$. Assume that the chain is hanging off the edge of the building, with the bottom of the chain at $x = 0$ and the top at $x = 3$.

Answer: 374.85 Joules

(2) A 3 meter chain with mass density $\rho(x) = 2x(4 - x)$ kg/m lies on the ground. Calculate the work required to lift the chain from the front end so that its bottom is 2 meters above the ground.

Solution: Consider a segment of the chain of length Δx that must be lifted x_j meters. The work needed to lift this segment is approximately

$$W_j \approx (\rho(x_j)\Delta x)9.8x_j \text{ Joules.}$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work needed to fully extend the chain is

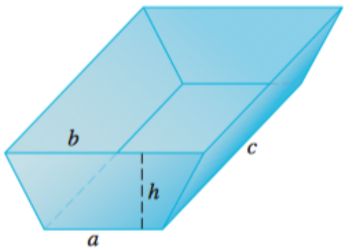
$$\int_0^3 9.8\rho(x)x\,dx = 9.8\int_0^3 (8x^2 - 2x^3)\,dx = 9.8\left(\frac{8}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^3 = 308.7 \text{ Joules.}$$

But we also need to lift the chain two meters off the ground after it's fully extended! This requires us to do work equal to 2 meters multiplied by the weight of the chain, which is

$$\int_0^3 9.8\rho(x)\,dx = 9.8\int_0^3 (8x - 2x^2)\,dx = 9.8\left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 176.4 \text{ Newtons.}$$

So lifting it another two meters after it's fully extended requires an additional 352.8 Joules of work. The total work is therefore 661.5 Joules.

(3) Calculate the work (in Joules) required to pump all of the water out of a trough as in the picture, where the water exits by pouring over the sides. Distances are in meters, and the density of water is 1000 kg/m^3.



SOLUTION: Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at height y meters is

$$w = a + \frac{y(b-a)}{h} \text{ meters,}$$

so the volume of each layer is

$$wc\Delta y = c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ meters}^3.$$

Thus, the force needed to lift a layer is

$$9800c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ Newtons.}$$

Each layer must be lifted $h - y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y(b-a)}{h}\right)dy = 9800c\left(\frac{ah^3}{3} + \frac{bh^2}{6}\right) \text{ Joules.}$$

§8.1 (Integration by Parts) Name:______________________
10 July 2018

(1) Evaluate the integral.

(a) $\int xe^{-x}\,dx$

Solution: Let $u = x$ and $dv = e^{-x}$. Then $u = x$, $du = dx$, and $v = -e^{-x}$. So

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int (1)(-e^{-x})\,dx = -e^{-x}(x+1) + C.$$

(b) $\int x^3 e^{x^2}\,dx.$

Solution: Let $w = x^2$. Then $dw = 2x\,dx$ and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw.$$

Now use integration by parts with $u = w$ and $dv = e^w$. We have $du = 1$ and $v = e^w$, so

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw = we^w - \int (1)e^w\,dw = we^w - e^2.$$

Finally, substitute back $w = x^2$ to get

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

(c) $\int_1^3 \ln x\,dx.$

Solution: Let $u = \ln x$ and $dv = 1$. Then $v = x$ and $du = 1/x$. So using integration by parts,

$$\int_1^3 \ln x\,dx = x\ln x\Big|_1^3 - \int_1^3 1\,dx = 3\ln 3 - 2.$$

(d) $\int xe^{2x}dx$

Solution: Integration by Parts gives us

$$\int xe^{2x}dx = x(1/2e^{2x}) - \int 1/2e^{2x}dx = 1/4e^{2x}(2x-1) + C$$

(e) $\int x^3 \ln x dx$

SOLUTION: Integration by Parts gives us

$$\int x^3 \ln x dx = \ln x \frac{1}{4}x^4 - \int \frac{1}{x}\frac{1}{4}x^4 dx = \frac{x^4}{16}(4\ln x - 1) + C$$

(f) $\int x \cos 2x dx$

SOLUTION: Using Integration by Parts, we get

$$\int x \cos 2x dx = x(\frac{1}{2}\sin 2x) - \int \frac{1}{2}\sin 2x dx = \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C$$

(g) $\int \frac{\ln x}{x^2} dx$

SOLUTION: Using Integration by Parts, we get

$$\int \frac{\ln x}{x^2} dx = \frac{-1}{x}\ln x - \int \frac{-1}{x^2} dx = \frac{-1}{x}(\ln x + 1) + C$$

(h) $\int \frac{\ln(\ln x)}{x} dx$

SOLUTION: Let $w = \ln x$. Then $dw = dx/x$, and we have

$$\int \frac{\ln(\ln x)}{x} dx = \int \ln w dw$$

Therefore we have

$$\int \frac{\ln(\ln x)}{x} dx = \ln x(\ln(\ln x)) - \ln x + C$$

(i) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{9+x^2}}dx$

SOLUTION: Let $u = 9 + x^2$. Then $du = 2x dx$, $x^2 2 = 9 - u$, and

$$\int_0^1 \frac{x^3}{\sqrt{9+x^2}}dx = \frac{1}{2}\int_9^{10}(u^{1/2} - 9u^{-1/2})du = 18 - \frac{17}{3}\sqrt{10}$$

(j) $\displaystyle\int x^4 e^{7x} dx$

SOLUTION: Let $u = 7x$. Then $du = 7dx$, and

$$\int x^4 e^{7x} dx = \frac{1}{7^5}\int u^4 e^u du$$

Applying integration by parts repeatedly, we would get

$$\frac{1}{7^5}\int u^4 e^u du = 7^{-5} e^u (u^4 - 4u^3 + 12u^2 - 24u + 24) + C$$

Plugging in $u = 7x$ gives the answer.

(k) $\displaystyle\int \frac{(\ln x)^2}{x^2} dx$

SOLUTION: Let $u = \ln(x)$. Then $du = \frac{1}{x} dx$, so the integral becomes

$$\int \frac{(\ln x)^2}{x^2} dx = \int \frac{u^2}{x} du.$$

To get rid of the x, use $u = \ln(x) \implies e^u = x$. We are now trying to integrate

$$\int u^2 e^{-u} du.$$

Use integration by parts twice:

$$\begin{aligned}\int u^2 e^{-u} du &= -2ue^{-u} + 2\int ue^{-u} du \\ &= -2ue^{-u} + 2\left(-ue^{-u} + \int e^{-u} dx\right) \\ &= -2ue^{-u} - 2ue^{-u} - 2e^{-u} + C \\ &= \frac{-4u - 2}{e^u} + C\end{aligned}$$

Put the original variable back in.

$$\int \frac{(\ln x)^2}{x^2} dx = \frac{-4\ln(x) - 2}{x} + C$$

(2) Find the volume of the solid obtained by revolving $y = \cos x$ for $0 \le x \le \pi/2$ around the y-axis.

SOLUTION: Using the cylindrical shells method, the volume V is given by

$$V = \int_a^b (2\pi r)h\,dx = 2\pi \int_0^{\pi/2} x\cos x\,dx.$$

and the radius $r = x$ varies from 0 to $\pi/2$, the height is $h = y = \cos x$. Then using integration by parts, with $u = x$ and $dv = \cos x$, we get

$$V = 2\pi \int_0^{\pi/2} x\cos x\,dx = 2\pi\,(x\sin x + \cos x)\Big|_0^{\pi/2} = \pi(\pi - 2).$$

(3) (a) Derive the reduction formula: $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

SOLUTION: Integrate the left hand side by parts. Set $u = x^n$, $dv = e^x\,dx$. Then $du = nx^{n-1}$ and $v = e^x$.

$$\int x^n e^x\,dx = x^n e^x - \int e^x n x^{n-1}\,dx$$

(b) Define functions $P_n(x)$ by the formula $\int x^n e^x\,dx = P_n(x)e^x$. Use the reduction formula from the previous part to prove that $P_n(x) = x^n - nP_{n-1}(x)$.

SOLUTION: We have

$$\begin{aligned} P_n(x)e^x &= \int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx \\ P_{n-1}(x)e^x &= \int x^{n-1}e^x\,dx \end{aligned}$$

Substituting the second line into the first, we have

$$P_n(x)e^x = x^n e^x - nP_{n-1}(x)e^x.$$

Factoring out e^x gives

$$P_n(x)e^x = (x^n - nP_{n-1}(x))\,e^x.$$

Dividing by e^x gives the formula we want.

(c) Use the recursion formula from the previous part to find $P_n(x)$ for $n = 0, 1, 2, 3, 4$.

SOLUTION:

$$\begin{aligned} &P_0(x) = 1 \\ &P_1(x) = x^1 - 1P_0(x) = x - 1 \\ &P_2(x) = x^2 - 2P_1(x) = x^2 - 2(x-1) = x^2 - 2x - 2 \\ &P_3(x) = x^3 - 3P_2(x) = x^3 - 3(x^2 - 2x - 2) = x^3 - 3x^2 - 6x - 6 \\ &P_4(x) = x^4 - 4P_3(x) = x^4 - 4(x^3 - 3x^2 - 6x - 6) = x^4 - 4x^3 - 12x^2 - 24x - 24 \end{aligned}$$