Math 1910, Prelim I

Name: Solutions

October 4th, 2016

PLACE AN X IN THE BOX TO INDICATE YOUR SECTION

☐	Aleksandra Niepla	201	MW 7:30–8:20pm	☐	Maru Sarazola	207	TR 9:05–9:55am
☐	Aleksandra Niepla	202	MW 8:35–9:25pm	☐	José Bastidas	208	TR 9:05–9:55am
☐	Maru Sarazola	203	TR 8:00–8:50am	☐	David Mehrle	209	TR 9:05–9:55am
☐	José Bastidas	204	TR 8:00–8:50am	☐	Nicholas LaVigne	210	TR 9:05–9:55am
☐	David Mehrle	205	TR 8:00–8:50am	☐	Abigail Turner	211	TR 12:20–1:10pm
☐	Nicholas LaVigne	206	TR 8:00–8:50am	☐	Abigail Turner	212	TR 1:25–2:15pm

INSTRUCTIONS

• PRINT your name and mark your section number **right now**.

• This test consists of eight pages (besides this cover sheet). Look over this test **as soon as the exam begins**. If you find any missing pages, please ask a proctor for another copy.

• **SHOW YOUR WORK.** To receive full credit, your answers must be neatly written, and logically organized. *Explain all steps that are not self–explanatory to an average student.* If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly.

• Scrap paper is available for rough work. You may not hand in work on scrap paper.

• You have 90 minutes to complete this exam.

• This is a closed book exam and no notes are allowed. You are **NOT** allowed to use a calculator, cell phone, or any other electronic devices.

• Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature of Student

OFFICIAL USE ONLY
(do not fill in)

1.	______	/ 30	______
2.	______	/ 30	______
3.	______	/ 20	______
4.	______	/ 30	______
5.	______	/ 20	______
6.	______	/ 20	______
Total:	______	/ 150	______

1. Compute the following indefinite integrals:

 a.) $\int \frac{x\cos x + \sqrt{x} + 1}{x}\,dx$

 We divide each term by x and then integrate them separately.

$$\begin{aligned}\int \frac{x\cos x + \sqrt{x} + 1}{x}\,dx &= \int \left(\cos x + x^{-1/2} + \frac{1}{x}\right) dx \\ &= \sin x + 2x^{1/2} + \ln(x) + C.\end{aligned}$$

 b.) $\int \frac{t+2}{\sqrt{t^2+4t}}\,dt$

 We make a substitution $u = t^2+4t$. Observe that $du = (2t+4)dt = 2(t+2)dt$. Therefore,

$$\int \frac{t+2}{\sqrt{t^2+4t}}\,dx = \int \frac{1}{\sqrt{t^2+4t}}(t+2)dt = \int u^{-1/2}\cdot\frac{1}{2}du = \frac{1}{2}\int u^{-1/2}du.$$

 So our integral is $u^{1/2} + C = (t^2+4t)^{1/2} + C$.

 c.) $\int \frac{e^x}{1+e^x}\,dx$

 We make a substitution $u = 1 + e^x$. We have $du = e^x dx$, so

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln(1+e^x) + C.$$

2. Compute the following definite integrals:

a.) $\int_0^{\pi} |\cos x|\, dx$

For a number x in the interval $[0, \pi]$, we have $|\cos x| = \cos x$ if $0 \le x \le \pi/2$ and $|\cos(x)| = -\cos(x)$ if $\pi/2 \le x \le \pi$. Therefore,

$$\int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx.$$

So our integral is equal to

$$\sin x\Big|_0^{\pi/2} - (-\sin x)\Big|_{\pi/2}^{\pi} = (\sin(\pi/2) - \sin(0)) + (-\sin(\pi) + \sin(\pi/2)) = (1-0) + (0+1) = 2.$$

b.) $\int_0^1 \frac{d}{dx}\left(\frac{x+x^2}{x^4+1}\right) dx$

The function $\frac{x+x^2}{x^4+1}$ is obviously an antiderivative of $\frac{d}{dx}\left(\frac{x+x^2}{x^4+1}\right)$. So we have

$$\int_0^1 \frac{d}{dx}\left(\frac{x+x^2}{x^4+1}\right) dx = \frac{x+x^2}{x^4+1}\Big|_0^1 = \frac{1+1^2}{1^4+1} - \frac{0+0^2}{0^4+1} = 2/2 - 0 = 1.$$

c.) $\int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos^2 x}\, dx$

Let us first compute $\int \frac{\sin x}{\cos^2 x}\, dx$. We make a substitution $u = \cos x$. We have $du = -\sin x\, dx$, so

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{-1}{u^2}\, du = -\int u^{-2}\, du = u^{-1} + C = (\cos x)^{-1} + C.$$

Therefore,

$$\int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos^2 x}\, dx = \frac{1}{\cos x}\Big|_{-\pi/4}^{\pi/4} = \frac{1}{\cos(\pi/4)} - \frac{1}{\cos(-\pi/4)}.$$

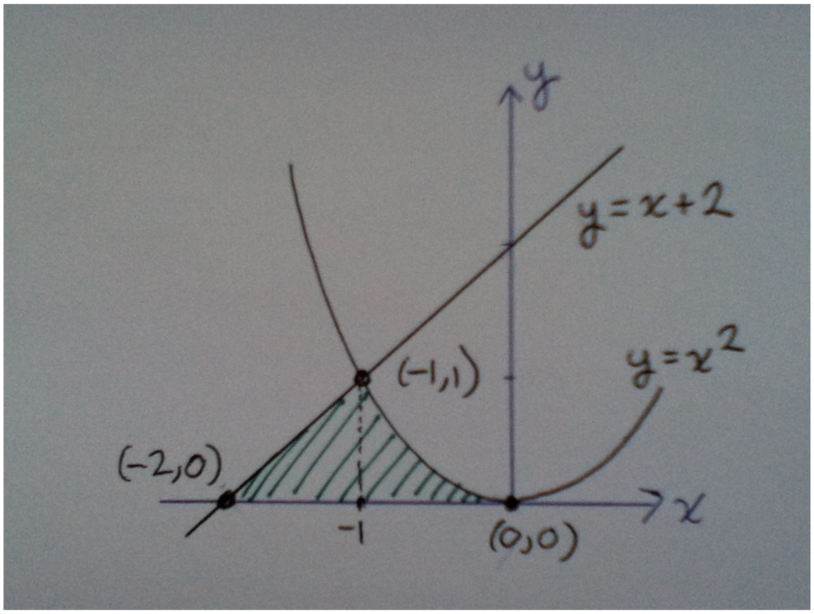
We have $\cos(\pi/4) = \cos(-\pi/4)$ since cosine is even (or since both values are equal to $\sqrt{2}/2$). Therefore, $\int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos^2 x}\, dx = 0$.

[One could have also proved this with no computations by noting that $\sin x/\cos^2 x$ is an odd function!]

3. Consider the "curved triangle" bounded above by the curves $y = x+2$ and $y = x^2$, and from below by the part of the x-axis that varies from $x = -2$ to 0. Draw the region and compute its area.

A picture of the "curved triangle" is given below. The curve $y = 0$ (i.e., the x-axis) intersects the curves $y = x + 2$ and $y = x^2$ at $(-2, 0)$ and $(0, 0)$, respectively.

To find where $y = x + 2$ and $y = x^2$ cross, we first solve $x + 2 = y = x^2$. So $0 = x^2 - x - 2 = (x-2)(x+1)$ and hence x is 2 or -1. Therefore, the two curves intersect at $(2, 4)$ and $(-1, 1)$. Only the point $(-1, 1)$ is relevant to our region.



The area of the region is then

$$A = \int_{-2}^{-1} (x + 2)\,dx + \int_{-1}^{0} x^2\,dx = (x^2/2 + 2x)\Big|_{-2}^{-1} + (x^3/3)\Big|_{-1}^{0}$$
$$= (1/2 - 2) - (2 - 4) + 0 - (-1/3) = 1/2 + 1/3 = 3/6 + 2/6 = 5/6.$$

Note that we broke up the area into two pieces (one from -2 to -1 where the line is on top and from -1 to 0 when the parabola is on top).

An alternate way to do the computation with a single integral is to see that as y varies from 0 to 1, the region is the one bounded above by $x = -\sqrt{y}$ (note the choice of sign!) and below by $x = y - 2$. The area is thus

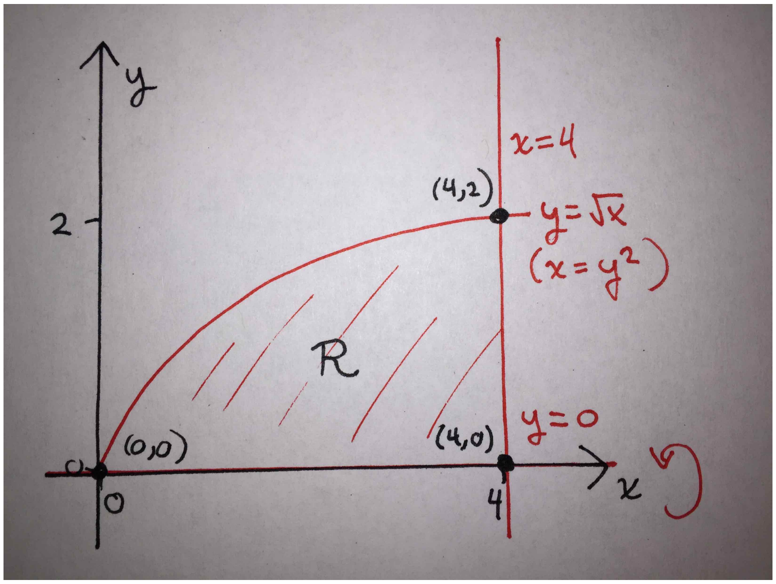
$$A = \int_0^1 -\sqrt{y} - (y - 2)\,dy = (-2/3 \cdot y^{3/2} - y^2/2 + 2y)\Big|_0^1$$
$$= (-2/3 - 1/2 + 2) - 0 = -4/6 - 3/6 + 12/6 = 5/6.$$

4. Let $\mathcal{R}$ be the region in the plane bounded by the curves:

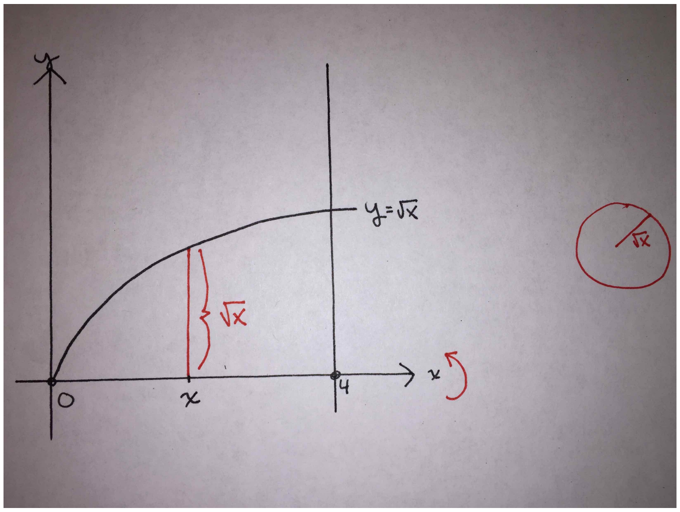
$$y = \sqrt{x}, \quad y = 0, \quad x = 4.$$

Let $\mathcal{S}$ be the solid generated by rotating the region $\mathcal{R}$ around the x-axis.

a.) Sketch the region $\mathcal{R}$.



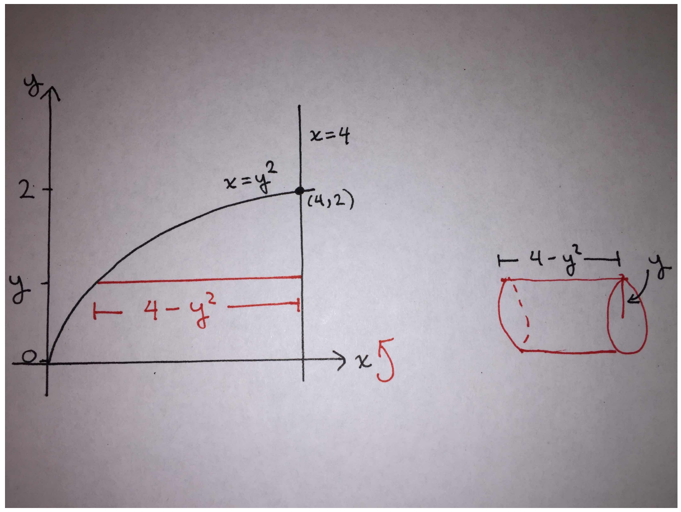
b.) Use the disk (or washer) method to find an integral that gives the volume of $\mathcal{S}$. Note: you do not need to compute the integral.



Using the disk method, the volume is:

$$\int_0^4 \pi(\sqrt{x})^2\, dx = \int_0^4 \pi x\, dx = \pi x^2/2\Big|_0^4 = \pi \cdot 4^2/2 = 8\pi.$$

c.) Use the shell method to find an integral that gives the volume of $\mathcal{S}$. Note: you do not need to compute the integral.



For a value $0 \leq y \leq 2$, the corresponding shell has radius y and length $4 - y^2$. Using the shell method, the volume is:

$$\begin{aligned}\int_0^2 2\pi \cdot y \cdot (4 - y^2)\, dy &= 8\pi \int_0^2 y\, dy - 2\pi \int_0^2 y^3\, dy \\ &= 4\pi y^2 \Big|_0^2 - \pi y^4/2 \Big|_0^2 \\ &= 4\pi \cdot 2^2 - \pi \cdot 2^4/2 \\ &= 16\pi - 8\pi = 8\pi.\end{aligned}$$

d.) Using one of the integrals from the previous parts, find the volume of $\mathcal{S}$.

Both integrals are computed above; they both give the value 8π.

5. a.) Compute the derivative of the function

$$g(x) = \int_x^{x^2} e^{t^2}\, dt.$$

Define the function $F(x) = \displaystyle\int_0^x e^{t^2}\, dt$. The Fundamental Theorem of Calculus says that

$$F'(x) = e^{x^2}.$$

We have

$$g(x) = \int_0^{x^2} e^{t^2}\, dt + \int_x^0 e^{t^2}\, dt = \int_0^{x^2} e^{t^2}\, dt - \int_0^x e^{t^2}\, dt = F(x^2) - F(x).$$

Taking derivatives (and using the chain rule), we have

$$g'(x) = F'(x^2) \cdot 2x - F'(x) = e^{(x^2)^2} \cdot 2x - e^{x^2} = 2xe^{x^4} - e^{x^2},$$

where in the last equality we have used $F'(x) = e^{x^2}$.

b.) Find the equation for the tangent line of the graph $y = g(x)$ at $x = 0$.

We have $g(0) = \int_0^0 e^{t^2}\, dt = 0$. Using the expression for $g'(x)$ in the previous question, we have

$$g'(0) = 2 \cdot 0 \cdot e^{0^4} - e^{0^2} = -1.$$

Since $g(0) = 0$ and $g'(0) = -1$, the equation for the tangent line is

$$y = -x;$$

it is the line of slope -1 that passes through $(0, 0)$.

6. a.) Verify that the following indefinite integral is correct:

$$\int e^x \sin(x)\,dx = \frac{1}{2}e^x \sin(x) - \frac{1}{2}e^x \cos(x) + C.$$

We take the derivative of the proposed antiderivative and show that it equals $e^x \sin(x)$.

$$\begin{aligned}\frac{d}{dx}\Big(\frac{1}{2}e^x \sin(x) - \frac{1}{2}e^x \cos(x)\Big) &= \big(\frac{1}{2}e^x \sin(x) + \frac{1}{2}e^x \cos(x)\big) - \big(\frac{1}{2}e^x \cos(x) - \frac{1}{2}e^x \sin(x)\big)\\ &= \frac{1}{2}e^x \sin(x) + \frac{1}{2}e^x \cos(x) - \frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x)\\ &= e^x \sin(x).\end{aligned}$$

b.) For each integer $N \geq 1$, define the number

$$R_N = \sum_{i=1}^{N} e^{i/N} \cdot \frac{1}{N}.$$

Compute the limit $\lim_{N\to\infty} R_N$ by recognizing R_N as a Riemann sum. Explain your answer.

First observe that the numbers $1/N, 2/N, \ldots, N/N = 1$ are the right endpoints of the subintervals obtained by cutting the interval $[0,1]$ into N subintervals with common width $\Delta x = 1/N$. We can then recognize R_N as the N-th right endpoint approximation to the integral

$$\int_0^1 e^x\,dx.$$

Therefore,

$$\lim_{N\to\infty} R_N = \int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1.$$

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