§8.7 (IMPROPER INTEGRALS) NAME: SOLUTIONS

18 July 2018

IMPROPER INTEGRALS

The **improper integral** of f over $[a, \infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \boxed{\lim_{R\to\infty} \int_a^R f(x)\,dx.} \quad (1)$$

We say that the improper integral **converges** if [the limit exists] (2), and **diverges** if [the limit does not exist] (3).

If $f(x)$ is continuous on $[a, b)$ with an infinite discontinuity at $x = b$, then the **improper integral** of f over $[a, b]$ is defined as:

$$\int_a^b f(x)\,dx = \boxed{\lim_{R\to b^-} \int_a^R f(x)\,dx.} \quad (4)$$

If $f(x)$ is continuous on $[a, b]$ and f has an infinite discontinuity at $f(x) = c$, where $a < c < b$, then the **improper integral** of f over the interval $[a, c]$ is defined as:

$$\int_a^b f(x)\,dx = \boxed{\lim_{R\to c^-} \int_a^R f(x)\,dx + \lim_{R\to c^+} \int_R^b f(x)\,dx.} \quad (5)$$

QUESTIONS

(1) Consider the integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$.

(a) Compute $\displaystyle\lim_{a\to\infty} \int_{-a}^{a} x\,dx$.

SOLUTION:

$$\lim_{a\to\infty} \int_{-a}^{a} x\,dx = \lim_{a\to\infty} \left.\frac{x^2}{2}\right|_{-a}^{a} = \lim_{a\to\infty} \left(\frac{a^2}{2} - \frac{(-a)^2}{2}\right) = 0.$$

(b) Is it fair to say that $\displaystyle\int_{-\infty}^{\infty} x\,dx$ converges? If not, then how should we define the improper integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$?

SOLUTION: The definition of the doubly infinite improper integral is

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx.$$

(2) Which of the following integrals is improper? Explain your answer but don't evaluate the integral.

(a) $\displaystyle\int_0^2 \frac{dx}{x^{1/3}}$

SOLUTION: Improper. The function $x^{-1/3}$ is not defined at 0.

(b) $\displaystyle\int_1^\infty \frac{dx}{x^{0.2}}$

SOLUTION: Improper. Infinite interval of integration.

(c) $\displaystyle\int_{-1}^\infty e^{-x}\,dx$

SOLUTION: Improper. Infinite interval of integration.

(d) $\displaystyle\int_0^1 e^{-x}\,dx$

SOLUTION: Proper. The function e^{-x} is continuous on the bounded interval $[0, 1]$.

(e) $\displaystyle\int_0^\pi \sec x\,dx$

SOLUTION: Improper. The function $\sec x$ is not defined at $\pi/2$.

(f) $\displaystyle\int_0^\infty \sin x\,dx$

SOLUTION: Improper. Infinite interval of integration.

(g) $\displaystyle\int_0^1 \sin x\,dx$

SOLUTION: Proper. The function $\sin x$is continuous on the bounded interval $[0, 1]$.

(h) $\displaystyle\int_0^1 \frac{dx}{\sqrt{3-x^2}}$

SOLUTION: Proper. The function $1/\sqrt{3-x^2}$ is continuous on the bounded interval $[0, 1]$.

(i) $\displaystyle\int_1^\infty \ln x\,dx$

SOLUTION: Improper. Infinite interval of integration.

(j) $\displaystyle\int_0^3 \ln x\,dx$

SOLUTION: Improper. The function $\ln x$ is not defined at 0.

(3) Determine whether the improper integral converges, and if it does, evaluate it.

(a) $\displaystyle\int_1^\infty \frac{1}{x^{20/19}}\,dx$

SOLUTION:

$$\begin{aligned}\int_1^\infty \frac{1}{x^{20/19}}\,dx &= \lim_{a\to\infty}\int_1^a \frac{1}{x^{20/19}}\,dx \\ &= \lim_{a\to\infty}\left(-19x^{-1/19}\right)\Big|_1^a \\ &= \lim_{a\to\infty}\left(-19-\frac{19}{a^{1/19}}\right) \\ &= 19-0=\boxed{19}\end{aligned}$$

(b) $\displaystyle\int_{20}^\infty \frac{1}{t}\,dt$

SOLUTION: The integral doesn't converge, because it's a p-integral with $p=1$.

(c) $\displaystyle\int_0^5 \frac{1}{x^{19/20}}\,dx$

SOLUTION: The function $x^{-19/20}$ is infinite at the endpoint zero, so it is improper.

$$\begin{aligned}\int_0^5 \frac{1}{x^{19/20}}\,dx &= \lim_{a\to 0}\int_a^5 \frac{1}{x^{19/20}} \\ &= \lim_{a\to 0}\left(20x^{1/20}\right)\Big|_a^5 \\ &= \lim_{a\to 0}\left(205^{1/20}-20a^{1/20}\right) \\ &= 20(5^{1/20}-0)=\boxed{20\cdot 5^{1/20}}\end{aligned}$$

(d) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$

SOLUTION: The function $f(x)=\frac{1}{\sqrt{3-x}}$ is infinite at $x=3$, so it is improper.

$$\begin{aligned}\int_1^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{a\to 3}\int_1^a \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{a\to 3}\left(2\sqrt{3-x}\right)\Big|_1^a \\ &= \lim_{a\to 3} 2\sqrt{3-a}-2\sqrt{2} \\ &= 2\sqrt{0}-2\sqrt{2}=\boxed{2\sqrt{2}}\end{aligned}$$

(e) $\displaystyle\int_{-2}^{4} \frac{1}{(x+2)^{1/3}}\,dx$

SOLUTION: The function $f(x) = (x+2)^{-1/3}$ is infinite at $x = -2$, so it is improper.

$$\begin{aligned}
\int_{-2}^{4} \frac{1}{(x+2)^{1/3}}\,dx &= \lim_{a\to -2} \int_{a}^{4} \frac{1}{(x+2)^{1/3}}\,dx \\
&= \lim_{a\to 2} \frac{3}{2}(x+2)^{2/3}\Big|_{a}^{4} \\
&= \lim_{a\to 2} \frac{3}{2}\left(6^{3/2} - (a+2)^{3/2}\right) \\
&= \frac{3}{2}\left(6^{2/3} - 0\right) = \boxed{\frac{3}{2}6^{2/3}}
\end{aligned}$$

The Comparison Test

The Comparison Test: Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then,

- If $\int_a^\infty \boxed{f(x)}^{(6)}\, dx$ converges, then $\int_a^\infty \boxed{g(x)}^{(7)}\, dx$ also converges.
- If $\int_a^\infty \boxed{g(x)}^{(8)}\, dx$ diverges, then $\int_a^\infty \boxed{f(x)}^{(9)}\, dx$ also diverges.

Most frequently, we compare integrals to the p-**integrals:**

- For $p > 1$: $\int_a^\infty \frac{1}{x^p}\, dx$ $\boxed{\text{converges}}^{(10)}$ and $\int_0^a \frac{1}{x^p}\, dx$ $\boxed{\text{diverges}}^{(11)}$.
- For $p < 1$: $\int_a^\infty \frac{1}{x^p}\, dx$ $\boxed{\text{diverges}}^{(12)}$ and $\int_0^a \frac{1}{x^p}\, dx$ $\boxed{\text{converges}}^{(13)}$.

Questions

(1) What happens when $p = 1$? Do the p-integrals $\int_a^\infty \frac{1}{x}\, dx$ and $\int_0^a \frac{1}{x}\, dx$ converge or diverge?

Solution:

$$\int_a^\infty \frac{1}{x}\, dx = \lim_{R\to\infty} \ln|R| - \ln|a| = \infty$$

$$\int_0^a \frac{1}{x}\, dx = \lim_{R\to 0} (\ln|a| - \ln|R|) = \infty$$

Both diverge.

(2) Show that $\int_1^\infty \frac{1}{\sqrt{x^4+1}}\, dx$ converges by comparing it with $\int_1^\infty x^{-2}\, dx$.

Solution: We first check that $\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}$: we know that $x^4 \leq x^4 + 1$, so $\sqrt{x^4} \leq \sqrt{x^4+1}$, or equivalently, $x^2 \leq \sqrt{x^4+1}$. Therefore,

$$\frac{1}{x^2} \geq \frac{1}{\sqrt{x^4+1}}.$$

So we can use the comparison test: if $\int_1^\infty x^{-2}\, dx$ converges, then so does $\int_1^\infty \frac{1}{\sqrt{x^4+1}}\, dx$. But the first one does because it's a p integral with $p > 1$.

(3) Determine whether the following integrals converge or diverge.

(a) $\int_1^\infty \frac{1-\sin x}{x^3+x}\,dx$

SOLUTION: Converges: $\frac{1-\sin x}{x^3+x} \leq \frac{2}{x^3+x} \leq \frac{2}{x^3}$ for every $x \in (1,\infty)$ and $\int_1^\infty \frac{2}{x^3}$ converges, so by the comparison theorem our original integral converges as well.

(b) $\int_0^1 \frac{e^x}{x^2}\,dx$

SOLUTION: Diverges; $\frac{e^x}{x^2} \geq \frac{1}{x^2}$ for every $x \in (0,1)$, and $\int_0^1 \frac{1}{x^2}$ diverges, so by the comparison theorem our original integral diverges as well.

(4) Show that $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$. Then use the comparison test to show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

SOLUTION: Note that $e^{-x^2} \leq e^{-x}$ for $x \in (1,\infty)$ (make sure you see why this is true) and

$$\int_1^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_1^R e^{-x}\,dx = \lim_{R\to\infty} -e^R + e^{-1} = e^{-1}.$$

Applying the comparison theorem we get that $\int_1^\infty e^{-x^2}\,dx$ converges.

Similarly, $e^{-x^2} \leq e^x$ for $x \in (-\infty,-1)$, and

$$\int_{-\infty}^{-1} e^x\,dx = e^{-1},$$

so applying the comparison theorem we get that $\int_{-\infty}^{-1} e^{-x^2}\,dx$ converges.

Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_1^\infty e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{-\infty}^{-1} e^{-x^2}\,dx$$

where the two improper integrals converge, and the middle integral is just a constant (since e^{-x^2} is continuous on the bounded interval $(-1,1)$).