§8.7 (IMPROPER INTEGRALS) NAME: ______________________

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IMPROPER INTEGRALS

The **improper integral** of f over $[a, \infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \boxed{} \quad (1)$$

We say that the improper integral **converges** if $\boxed{}$ (2), and **diverges** if $\boxed{}$ (3).

If $f(x)$ is continuous on $[a, b)$ with an infinite discontinuity at $x = b$, then the **improper integral** of f over $[a, b]$ is defined as:

$$\int_a^b f(x)\,dx = \boxed{} \quad (4)$$

If $f(x)$ is continuous on $[a, b]$ and f has an infinite discontinuity at $f(x) = c$, where $a < c < b$, then the **improper integral** of f over the interval $[a, c]$ is defined as:

$$\int_a^b f(x)\,dx = \boxed{} \quad (5)$$

QUESTIONS

(1) Consider the integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$.

(a) Compute $\displaystyle\lim_{a\to\infty} \int_{-a}^{a} x\,dx$.

(b) Is it fair to say that $\displaystyle\int_{-\infty}^{\infty} x\,dx$ converges? If not, then how should we define the improper integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$?

(2) Which of the following integrals is improper? Explain your answer but don't evaluate the integral.

(a) $\displaystyle\int_0^2 \frac{dx}{x^{1/3}}$

(b) $\displaystyle\int_1^\infty \frac{dx}{x^{0.2}}$

(c) $\displaystyle\int_{-1}^\infty e^{-x}\,dx$

(d) $\displaystyle\int_0^1 e^{-x}\,dx$

(e) $\displaystyle\int_0^\pi \sec x\,dx$

(f) $\displaystyle\int_0^\infty \sin x\,dx$

(g) $\displaystyle\int_0^1 \sin x\,dx$

(h) $\displaystyle\int_0^1 \frac{dx}{\sqrt{3-x^2}}$

(i) $\displaystyle\int_1^\infty \ln x\,dx$

(j) $\displaystyle\int_0^3 \ln x\,dx$

(3) Determine whether the improper integral converges, and if it does, evaluate it.

(a) $\displaystyle\int_1^\infty \frac{1}{x^{20/19}}\,dx$

(b) $\displaystyle\int_{20}^\infty \frac{1}{t}\,dt$

(c) $\displaystyle\int_0^5 \frac{1}{x^{19/20}}\,dx$

(d) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$

(e) $\displaystyle\int_{-2}^4 \frac{1}{(x+2)^{1/3}}\,dx$

THE COMPARISON TEST

The Comparison Test: Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then,

- If $\int_a^\infty \boxed{}^{(6)}\,dx$ converges, then $\int_a^\infty \boxed{}^{(7)}\,dx$ also converges.
- If $\int_a^\infty \boxed{}^{(8)}\,dx$ diverges, then $\int_a^\infty \boxed{}^{(9)}\,dx$ also diverges.

Most frequently, we compare integrals to the p-**integrals:**

- For $p > 1$: $\int_a^\infty \frac{1}{x^p}\,dx$ $\boxed{}^{(10)}$ and $\int_0^a \frac{1}{x^p}\,dx$ $\boxed{}^{(11)}$.
- For $p < 1$: $\int_a^\infty \frac{1}{x^p}\,dx$ $\boxed{}^{(12)}$ and $\int_0^a \frac{1}{x^p}\,dx$ $\boxed{}^{(13)}$.

QUESTIONS

(1) What happens when $p = 1$? Do the p-integrals $\int_a^\infty \frac{1}{x}\,dx$ and $\int_0^a \frac{1}{x}\,dx$ converge or diverge?

(2) Show that $\int_1^\infty \frac{1}{\sqrt{x^4+1}}\,dx$ converges by comparing it with $\int_1^\infty x^{-2}\,dx$.

(3) Determine whether the following integrals converge or diverge.

(a) $\displaystyle\int_1^\infty \frac{1-\sin x}{x^3+x}\,dx$

(b) $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$

(4) Show that $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$. Then use the comparison test to show that $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx$ converges.